Comparison of Stability of Selected Numerical Methods for Solving Stiff Semi-Linear Differential Equations

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Abstract

Attention was focused on the explicit Exponential Time Differencing (ETD) integrators that are designed to solve stiff semi-linear problems. Semi-linear PDEs can be split into a linear part, which contains the stiffest part of the dynamics of the problem, and a nonlinear part, which varies more slowly than the linear part. The ETD methods solve the linear part exactly, and then explicitly approximate the remaining part by polynomial approximations. The research involves an analytical examination and comparison of the asymptotic stability properties of some Exponential Time Differencing Schemes (ETD1, ETD2, ETD2RK1 and ETD2RK2) methods in order to present the advantage of these methods in overcoming the stability constraints.

Keywords : Differential Equation, Exponential Time Differencing Method, Exponential Time Differencing Runge Kutta Method, Integrating Factor Method, Ordinary differential Equation, Partial Differential Equation and Stiff linear differential Equation.

Introduction

Various problems in the world can be solved when they are modeled and presented in the form of an ordinary differential equation or partial differential equation. However, there are times where different phenomena acting on very different time scales occur simultaneously introducing a parameter called stiff parameter which sometimes makes it difficult to solve. All differential equations with this property are said to be stiff differential equations. According to Curtiss, et al (1952) the earliest detection of stiffness in differential equations in the digital computer era, was apparently far in advance of its time. They named the phenomenon and spotted the nature of stiffness (stability requirement dictates the choice of the step size to be very small).To resolve the problem they recommended possible methods such as the Backward Differentiation Formula for numerical integration. This study looked at how to solve stiff differential equations using the Exponential Time Differencing Schemes making reference to their asymptotic stabilities.

Related Works

In 1963, Dahlquist defined the stiff problem and demonstrated the difficulties that standard differential equation solvers have with stiff differential equations. Dahlquist et al (1973), defined a stiff system as one containing very fast components as well as very slow components; they represent coupled physical systems having components varying with very different time scales: that is, they are systems having some components varying much more rapidly than the others. Another significant person worth mentioning in this field of studies is Gear. He made considerable efforts to develop numerical integration for stiff problems and hence brought the attention of the mathematical and computer science community to stiff problems.(Gear,1968). Hairer, et al (1996) studied different methods of solving differential equations and identified that all the explicit methods were unable to solve a particular type of differential equation (called stiff differential equation). They defined stiff equation as a problem for which explicit methods don't work.

According to Du (2004), Exponential Time Differencing schemes are time integration methods that can be efficiently combined with spatial spectral approximations to provide very high resolution to the smooth solutions of some linear and nonlinear partial differential equations. Du, explained in his paper the stability properties of some exponential time differencing schemes, he also presented their application to the numerical solution of the scalar Allen-Cahn equation in two and three dimensional spaces. Livermore et al (2007) explained that over the last decade there has been renewed interest in applying exponential time differencing (ETD) schemes to the solution of stiff systems. He presented an implementation of such a scheme to the fully spectral solution of the incompressible magneto hydrodynamic equations in a spherical shell. Quit recently, Hala have carried out some research on the stability of numerical methods. According to Hala (2008), the stability of a given method for solving a system of ODE is a theoretical measure of the extent to which the method produces satisfactory approximation. According to him, stability is related to the accuracy of the methods and are referred to as errors not growing in subsequent steps.

According to Thohura, et al (2013), although a number of methods have been developed and many more basic formulas suggested for stiff equations, until recently there has been little advice or guidance to help a practitioner choose a good method for this problem. In case of stiff differential equations, stability requirements force the solver to take a lot of small time steps; this happens when we have a system of coupled differential equations that have two or more very different scales of the independent variable over which we are integrated. Various authors have looked at solutions of stiff differential equations using the Exponential Time Differencing (ETD) and Exponential Time Differencing Runge-Kutta (ETDRK) methods without checking their stability. This study seeks to compare the asymptotic stability of some of these methods and to obtain stability expressions for these numerical schemes.

Model Formulation

To derive the s-step ETD schemes, consider for simplicity a single model of stiff ODE

$$\frac{du(t)}{dt} = cu(t) + F(u(t), t) \qquad (1)$$

we multiply equation (1) through by the integrating factor e^{-ct}, and then integrate the equation over a single time step from $t = t_n$ to $t = t_{n+1} = t_n + \Delta t$ to get

$$u(t_{n+1}) = u(t_n)e^{c\Delta t} + e^{c\Delta t}\int_0^{\Delta t} e^{-c\tau} F(u(t_n+\tau), t_n+\tau)d\tau \qquad (2)$$

This formula is exact, and the next step is to derive approximations to the integral in equation (2). This procedure does not introduce an unwanted fast time scale into the solution and the schemes can be generalized to arbitrary order.

If we apply the Newton Backward Difference Formula, using information about $F(u(t), t)$ at the nth and previous time steps, we can write a polynomial approximation to $F(u(t_n+\tau), t_n+\tau)$ in the form

$$F(u(t_n+\tau), t_n+\tau) \approx G_n(t_n+\tau) = \sum_{m=0}^{s-1}(-1)^m \binom{-\tau/\Delta t}{m} \nabla^m G_n(t_n), \qquad (3)$$

where ∇ is the backward difference operator defined as follows

$$\nabla^m G_n(t_n) = \sum_{k=0}^{m}(-1)^k \binom{m}{k} G_{n-k}(t_{n-k}),$$

$$\approx \sum_{k=0}^{m}(-1)^k \binom{m}{k} F(u(t_{n-k}), t_{n-k}), \qquad (4)$$

and $m!\binom{-q}{m} = (-q)(-q-1)\ldots(-q-m+1), m = 1, \ldots\ldots, s-1$

(note that $0!\binom{-q}{0} = 1$. If we substitute the approximation equation (3) in the integrand equation (2), we get

$$u(t_{n+1}) - u(t_n)e^{c\Delta t} \approx \Delta t \sum_{m=0}^{s-1}(-1)^m \int_0^1 e^{c\Delta t(1-q)} \binom{-q}{m} dq \nabla^m G_n(t_n) \qquad (5)$$

Where $q = \tau/\Delta t$.

We will indicate the integral in equation (5) by

$$g_m = (-1)^m \int_0^1 e^{c\Delta t(1-q)} \binom{-q}{m} dq, \qquad (6)$$

and then calculate the g_m by bringing in the generating function. For $z \in R, |z| < 1$, we define the generating function

$$\Gamma(z) = \sum_{m=0}^{\infty} g_m z^m,$$

$$= \int_0^1 e^{c\Delta t(1-q)} \sum_{m=0}^{\infty} \binom{-q}{m} (-z)^m \, dq,$$

$$= \int_0^1 e^{c\Delta t(1-q)} (1-z)^{-q} dq,$$

$$= \frac{e^{c\Delta t}\left(1 - z - e^{-c\Delta t}\right)}{(1-z)\left(c\Delta t + log(1-z)\right)} \qquad (7)$$

Rearranging equation (7) to form

$$\left(c\Delta t + log(1-z)\right)\Gamma(z) = e^{c\Delta t} - (1-z)^{-1},$$

and expanding as a power series in z

$$\left(c\Delta t - z - \frac{z^2}{2} - \frac{z^3}{3} - \cdots\right)(g_0 + g_1 z + g_2 z^2 + \cdots) = e^{c\Delta t} - 1 - z - z^2 - z^3 - \cdots,$$

we can find a recurrence relation for the g_m for $m \geq 0$ by equating like powers of z

$$c\Delta t g_o = e^{c\Delta t} - 1, \qquad (7a)$$

$$c\Delta t g_{m+1} + 1 = g_m + \frac{1}{2} g_{m-1} + \frac{1}{3} g_{m-2} + \cdots + \frac{1}{m+1} g_0 = \sum_{k=0}^{m} \frac{1}{m+1-k} g_k \qquad (8)$$

Haven determined the $g_{m,}$ the ETD schemes equation (5) then can be given in explicit forms.
Substituting equation (4) and equation (6) in equation (5), we deduce the general generating formula of ETD schemes of order *s*

$$u_{n+1} = u_n e^{c\Delta t} + \Delta t \sum_{m=0}^{s-1} g_m \sum_{k=0}^{m} (-1)^k \binom{m}{k} F_{n-k} \qquad (9)$$

where u_n and F_n denote the numerical approximation to u(t_n and F(u(t_n, t_n respectively, and the g_m are given by equation (8).

ETD Schemes

ETD1 Scheme

From equation (7a) above, g_0 can be written as

$$g_0 = \frac{e^{c\Delta t} - 1}{c\Delta t}$$

To obtain the ETD1 scheme, we set $s = 1$ in the explicit generating formula equation (9) to get

$$u_{n+1} = u_n e^{c\Delta t} + \Delta t g_0 F_n$$

$u_{n+1} = u_n e^{c\Delta t} + \Delta t \left(\frac{e^{c\Delta t}-1}{c\Delta t}\right) F_n$, hence the ETD1 scheme is given by;

$$u_{n+1} = u_n e^{c\Delta t} + \left(e^{c\Delta t} - 1\right) F_n / c, \qquad (10)$$

ETD2 Scheme

In the same manner, setting $s = 2$ in equation (9) gives us the second-order ETD2 scheme

$$u_{n+1} = u_n e^{c\Delta t} + \frac{\left\{\left((c\Delta t + 1)e^{c\Delta t} - 2c\Delta t - 1\right) F_n + \left(-e^{c\Delta t} + c\Delta t + 1\right) F_{n-1}\right\}}{c^2 \Delta t} \qquad (11)$$

Generally, for the one-step time-discretization methods and the Runge-Kutta (RK) methods, all the information required to start the integration is available. However, for the multi-step time-discretization methods this is not true.

These methods require the evaluations of a certain number of starting values of the nonlinear term $F(u(t),t)$at the nth and previous time steps to build the history required for the calculations. Therefore, it is desirable to construct ETD methods that are based on RK methods.

ETD Runge - Kutta Schemes

Cox et al (2002), constructed a second-order ETD Runge-Kutta method, analogous to the "improved Euler" method given as follows.

ETDRK1 Scheme

Putting $s = 1$ in equation (9) gives

$$u_{n+1} = u_n e^{c\Delta t} + \left(e^{c\Delta t} - 1\right) F_n/c. \qquad (12)$$

Let $a_n \approx u_{n+1}$, than it implies that

$$a_n = u_n e^{c\Delta t} + \frac{\left(e^{c\Delta t} - 1\right)F_n}{c} \qquad (13)$$

The term$a_n$approximates the value of u at $t_n + \Delta t$. The next step is to approximate F in the interval $t_n \leq t \leq t_{n+1}$, with

$$F = F_n + (t - t_n)\,(F(a_n, t_n + \Delta t) - F_n)/\Delta t + O(\Delta t^2)$$

and substitute into equation (13) to give the ETD2RK1 scheme

$$u_{n+1} = a_n + \left(e^{c\Delta t} - c\Delta t - 1\right)(F(a_n, t_n + \Delta t) - F_n)/(c^2 \Delta t). \qquad (14)$$

ETD2RK2 Scheme

In a similar way, we can also form an ETD2RK2 scheme analogous to the "modified Euler" method. The first step

$$a_n = u_n e^{c\Delta t/2} + \left(e^{\frac{c\Delta t}{2}} - 1\right) F_n/c,$$

is formed by taking half a step of equation (13); then use the approximation

$F = F_n + \frac{(t-t_n)}{\Delta t/2}(F(a_n, t_n + \Delta t/2) - F_n) + O(\Delta t^2),$

in the interval $[t_n, t_n + \Delta t]$ in equation (2) to deduce the ETD2RK2 scheme

$$u_{n+1} = u_n e^{c\Delta t} + \left\{\left((c\Delta t - 2)e^{c\Delta t} + c\Delta t + 2\right)F_n + 2\left(e^{c\Delta t} - c\Delta t - 1\right)F(a_n, t_n + \Delta t/2)\right\}/c^2\,\Delta t \qquad (15)$$

Finally, we note that as $c \to 0$ in the coefficients of the s-order ETD-RK methods, the methods reduce to the corresponding order of the Runge-Kutta schemes.

Stability Analysis

The approach developed by Beylkin, et al (1998) studied the stability for a family of explicit and implicit ELP schemes, and showed that these schemes have significantly better stability properties when compared with known implicit-explicit schemes. The approach developed for the stability analysis of composite schemes, i.e. schemes that use different methods for the linear and nonlinear parts of the equation, computes the boundaries of the stability regions for a general test problem. That is to analyze the stability of the ETD schemes, we linearize the autonomous ODE

$$\frac{du(t)}{dt} = cu(t) + F\big(u(t)\big), \qquad (16)$$

about a fixed point u_0 (so that $cu_0 + F(u_0) = 0$, to obtain

$$\frac{du(t)}{dt} = cu(t) + \lambda u(t) = (c+\lambda)u(t), \qquad (17)$$

where $u(t)$ is the perturbation to u_0 and

$$\lambda = \frac{dF\big(u(t)\big)}{du}|u(t) = u_0$$

Again $F(u_0) - F(u_n) = \lambda(u_0) - \lambda(u_n)$

But $u_0 = 0$

$$\text{Therefore, } F(u_n) = \lambda u_n \qquad (18)$$

In order to keep the fixed point u_0 stable, we require $\Re(c+\lambda)<0$ (note that the fixed points of the ETD methods are the same as those of the ODE equation (16), in contrast to the IF methods which do not preserve the fixed points for the ODE that they discretize. It seems desirable for a numerical method to fulfill this property with respect to capturing as much of the dynamics of the system as possible). If both c and λ are complex, the stability region is four-dimensional. But if both c and λ are pure imaginary or pure real, or if λ is complex and c is fixed and real then the stability region is two-dimensional. This study concentrates on two cases to determine whether the schemes are asymptotically stable. The conditions are as follows;

- c is fixed and negative andλ and c are purely real.
- c is negative and both c and λ are purely real.

Algorithm

To determine whether an exponential time differencing method is asymptotically stable, considering the problem

$$\frac{du(t)}{dt}=cu(t)+\lambda u(t)$$

Step 1: Solve the problem using any one of the ETD1, ETD2, ETD2RK2 and ETD2RK2 methods.
Step 2: Divide through the u_{n+1} solution with u_n to obtain an equation for $\frac{u_{n+1}}{u_n}$
Step 3: Set $r=\frac{u_{n+1}}{u_n}$, $x=\lambda\Delta t$ and $y=c\Delta t$, where c and λ are parameters in the given problem and Δt is the time step.
Step 4: For a scheme to be asymptotically stable then;

$$r=\frac{u_{n+1}}{u_n}\le 1$$

Given the problem above, the asymptotic stability of the schemes can be determined as follows;

Stability of ETD1 Scheme

Equation (10) can be written in the form;

$$\frac{u_{n+1}}{u_n}=e^{c\Delta t}+\frac{\left(e^{c\Delta t}-1\right)}{cu_n}F_n \qquad (19)$$

From equation (18), F_n can be written as;

$$F_n=\lambda u_n \qquad (20)$$

Putting equation (20) in to equation (19), gives;

$$\frac{u_{n+1}}{u_n}=e^{c\Delta t}+\frac{\left(e^{c\Delta t}-1\right)}{cu_n}\lambda u_n$$

$$\frac{u_{n+1}}{u_n}=e^{c\Delta t}+\frac{\left(e^{c\Delta t}-1\right)}{c}\lambda \qquad (21)$$

putting $x=\lambda\Delta t, y=c\Delta t$ and$r=\frac{u_{n+1}}{u_n}$ in to the above equation gives;

$$r=e^y+\frac{x}{y}(e^y-1) \qquad (22)$$

If

$$r=e^y+\frac{x}{y}(e^y-1)\le 1 \qquad (23)$$

then ETD1 is asymptotically stable.

Stability of ETD2 Scheme

Equation (11) can be written in the form;

$$\frac{u_{n+1}}{u_n}=e^{c\Delta t}+\frac{\left\{\left((c\Delta t+1)e^{c\Delta t}-2c\Delta t-1\right)F_n+\left(-e^{c\Delta t}+c\Delta t+1\right)F_{n-1}\right\}}{c^2\Delta t u_n}$$

Substituting equation (18) in to the above equation gives.

$$\frac{u_{n+1}}{u_n}=e^{c\Delta t}+\frac{\left\{\left((c\Delta t+1)e^{c\Delta t}-2c\Delta t-1\right)\lambda u_n+\left(-e^{c\Delta t}+c\Delta t+1\right)F_{n-1}\right\}}{c^2\Delta t u_n} \qquad (24)$$

putting $x=\lambda\Delta t, y=c\Delta t$ and$r=\frac{u_{n+1}}{u_n}$ in to the above equation gives;

$$y^2r^2-(y^2e^y+[(y+1)e^y-2y-1]x)r+(e^y-y-1)x=0$$

$$r=e^y+\left(\frac{e^y-1}{y}\right)x+\left(\frac{(e^y-y-1)}{y^2}\right)x^2 \qquad (25)$$

If

$$r=e^y+\left(\frac{e^y-1}{y}\right)x+\left(\frac{(e^y-y-1)}{y^2}\right)x^2\leq 1 \qquad (26)$$

then ETD2 is asymptotically stable. Stability of ETD2RK1 Scheme Equation (14) can be written as;

$$F\,(a_n,t_n+\Delta t)/c^2\,\Delta t u_n$$

$$\frac{u_{n+1}}{u_n}=e^{c\Delta t}+\frac{\left(e^{c\Delta t}-1\right)F_n}{cu_n}+\left(e^{c\Delta t}-c\Delta t-1\right)$$

Substituting $F_n=\lambda u_n$ in to the above equation gives

$$\frac{u_{n+1}}{u_n}=e^{c\Delta t}+\frac{\left(e^{c\Delta t}-1\right)\lambda u_n}{cu_n}+\left(\left(e^{c\Delta t}-c\Delta t-1\right)\frac{F(a_n,t_n+\Delta t)}{c^2\Delta t u_n}\right) \qquad (27)$$

Putting $x=\lambda\Delta t, y=c\Delta t$ and$r=\frac{u_{n+1}}{u_n}$ in to the above equation, we get

$$r=e^y+(e^y-1)\frac{\lambda u_n}{cu_n}+\left(\frac{(e^y-y-1)(e^y-1)}{y^2}\right)x^2$$

$$r=e^y+(e^y-1)\frac{x}{y}+\left(\frac{(e^y-y-1)(e^y-1)}{y^2}\right)x^2$$

$$r=e^y+\left(\frac{e^y-1}{y}\right)x+\left(\frac{(e^y-y-1)(e^y-1)}{y^2}\right)x^2 \qquad (28)$$

If

$$r=e^y+\left(\frac{e^y-1}{y}\right)x+\left(\frac{(e^y-y-1)(e^y-1)}{y^2}\right)x^2\leq 1 \qquad (29)$$

then ETD2RK2 is asymptotically stable. Stability of ETD2RK2 Scheme Equation (15) can be written as

$$\frac{u_{n+1}}{u_n}=e^{c\Delta t}+\frac{\left\{\left((c\Delta t-2)e^{c\Delta t}+c\Delta t+2\right)F_n+2\left(e^{c\Delta t}-c\Delta t-1\right)F(a_n,t_n+\Delta t)+\Delta t/2)\right\}}{u_n c^2\Delta t} \qquad (30)$$

Let $r=\frac{u_{n+1}}{u_n}, x=\lambda\Delta t$ and $y=c\Delta t$

$$r=e^y+\left(\frac{2(e^y-y-1)e^{y/2}+(y-2)e^y+y+2}{y^2}\right)x+\left(\frac{2(e^y-y-1)\left(e^{y/2}-1\right)}{y^3}\right)x^2 \qquad (31)$$

If

$$r=e^y+\left(\frac{2(e^y-y-1)e^{y/2}+(y-2)e^y+y+2}{y^2}\right)x+\left(\frac{2(e^y-y-1)\left(e^{y/2}-1\right)}{y^3}\right)x^2\leq 1 \qquad (32)$$

then ETD2RK2 is asymptotically stable.

Results

Computational Results of Stability of ETD Schemes

Du et al (2004), gave the parameter values for c, λ and Δt. These values were adopted in this study to compute the values of *x* and *y* given that $x=\lambda\Delta t\wedge y=c\Delta t$.

Following the first condition under the stability analysis, where λ is real and c is fixed, negative and both $\lambda \wedge c$ are purely real; the values of x and y were computed using the adopted values for the parameters $c, \lambda \wedge \Delta t$ and represented in a tabular form below.

Table 1: x and y Values Given that c is Fixed and Negative and λ and c are both Purely Real.

Δt	c	λ	x	y
1×10^{-3}	-0.1	1×10^{-4}	1×10^{-7}	-1×10^{-4}
2×10^{-3}	-0.1	1×10^{-5}	2×10^{-8}	-2×10^{-4}
3×10^{-3}	-0.1	1×10^{-6}	3×10^{-9}	-3×10^{-4}
4×10^{-3}	-0.1	1×10^{-7}	4×10^{-10}	-4×10^{-4}
5×10^{-3}	-0.1	1×10^{-8}	5×10^{-11}	-5×10^{-4}
6×10^{-3}	-0.1	1×10^{-9}	6×10^{-12}	-6×10^{-4}

It can be observed from Table (1) above that as the values of Δt andλ increase and c remain constant, values of x and y decreases accordingly. Because of the negative values of c, all values obtained for y were also negative. From tables (1), the computed values of *x* and *y* were used to carry out the computations for the r values for ETDI, ETD2, ETD2RK1 and ETD2RK2. Considering the condition that c is fixed and negative and both λ and c are purely real, equations (17), equation (21) and equation (22) computes the r values for ETD1, in a similar way, equations (17), equation (24) and equation (25) computes the r values for ETD2, while equations (17), equation (27) and equation (28) computes the r values for ETD2RK1. Finally equations (17), equation (30) and equation (31) computes the *r* values for ETD2RK2. Below is a summary of the computed values of *r* for the schemes.

Table 2: The *r* Values of the Schemes when Parameter c is Fixed and Negative and λ is Complex

$\lvert r \rvert VALUESOFSCHEMES$			
$ETD1$	$ETD2$	$ETD2RK1$	$ETD2RK2$
0.9999	0.9999	0.9999	0.9999
0.9998	0.9998	0.9998	0.9998
0.9997	0.9997	0.9997	0.9997
0.9996	0.9996	0.9996	0.9996
0.9995	0.9995	0.9995	0.9995
0.9994	0.9994	0.9994	0.9994

From Table (2), all values corresponding to the ETD and ETDRK schemes are less than one indicating that all schemes are asymptotically stable at these points studied. It can also be observed that at each Δt all schemes have the same values and this is true for all values of Δt. Hence none of the schemes can be said to be more asymptotically stable than the other. Again descending down the table, the values of r corresponding to the schemes decreases, hence making the schemes more stable. Considering the second condition under the stability analysis, where c is changing and negative and both $\lambda \wedge c$ are purely real; the values of x and y were computed using the adopted values for the parameters $c, \lambda \wedge \Delta t$ and represented in a tabular below.

Table 3: x and y Values Given c is changing and Negative and λ and c are Real

Δt	c	λ	x	y
1×10^{-3}	-0.1	1×10^{-4}	1×10^{-7}	-1×10^{-4}
2×10^{-3}	-0.2	1×10^{-5}	2×10^{-8}	-4×10^{-4}
3×10^{-3}	-0.3	1×10^{-6}	3×10^{-9}	-9×10^{-4}
4×10^{-3}	-0.4	1×10^{-7}	4×10^{-10}	-1.6×10^{-3}
5×10^{-3}	-0.5	1×10^{-8}	5×10^{-11}	-2.5×10^{-3}
6×10^{-3}	-0.6	1×10^{-9}	6×10^{-12}	-3.6×10^{-3}

Values from Table (3) show that given the condition that c is changing and negative and both $c \wedge \lambda$ are real, values of x and y decrease as Δt increases. Considering the condition that c is changing and negative and both λ and c are purely real, equations (17), equation (21) and equation (22) computes the r values for ETD1, in a similar way, equations (17), equation (24) and equation (25) computes the r values for ETD2, while equations (17), equation (27) and equation (28) computes the r values for ETD2RK1. Finally equations (17), (30) and (31) computes the *r* values for ETD2RK2. Below is a summary of the computed values of r for the schemes.

Table 4.0: The *r* Values of the Schemes when Parameter c is changing and Negative and λ is Complex.

$\|r\| VALUESOFSCHEMES$			
$ETD1$	$ETD2$	$ETD2RK1$	$ETD2RK2$
0.9999	0.9999	0.9999	0.9999
0.9996	0.9996	0.9996	0.9996
0.9991	0.9991	0.9991	0.9991
0.9980401	0.9980401	0.9980401	0.9980401
0.997503	0.997503	0.997503	0.997503
0.996406	0.996406	0.996406	0.996406

Again in Table (4), all values corresponding to ETD and ETDRK schemes are less than one, indicating that all schemes are asymptotically stable at these points. It can also be observed that at each Δt all the schemes have the same values suggesting that none of the schemes studied is better in terms of asymptotic stability than the other. Descending down the table, the r values of the schemes are decreasing, hence suggesting that the schemes are becoming more asymptotically stable.

Discussion

From Table (1), given that $1 \times 10^{-3} \leq \Delta t \leq 6 \times 10^{-3}$, c is fixed and negative and λ is complex, results obtained for $x = \lambda\Delta t$ and $y = c\Delta t$ showed that both values of x and y increased for every increase in Δt, however all values of y were negative. In Table (2), if c is changing and negative and λ is real, the values of x remained the same while y values were found to be changing. From Table (3), all values corresponding to the various ETD and ETDRK schemes are less than one indicating that all schemes are asymptotically stable at these points studied. It can also be observed that at each Δt all schemes have the same values and this is true for all values of Δt. Hence none of the schemes can be said to be more asymptotically stable than the order. Again descending down the table, the values of r corresponding to the schemes decreases, hence making the schemes more stable. Again in Table (4), all values corresponding to ETD and ETDRK schemes are less than one, indicating that all schemes are asymptotically stable at these points. It can also be observed that at each Δt all the schemes have the same values suggesting that none of the schemes studied is better in terms of asymptotic stability than the other. Descending down the table, the r values of the schemes are decreasing, hence suggesting that the schemes are becoming more asymptotically stable.

Conclusion

This research suggest that the comparison of the asymptotic stability of ETD1, ETD2, ETD2RK1 and ETD2RK2 schemes in solving the stiff semi-linear differential equation (17) was properly executed. This was made possible when some parameters $c, \lambda \wedge \Delta t$ were adopted and used for computations. To ensure that the first objective was met, ETD1, ETD2, ETD2RK1 and ETD2RK2 schemes were used to solve the stiff semi-linear differential equation (17) to obtain the asymptotic stability expressions; equation (23), equation (24), equation (29) and equation (32). The second objective suggested the following conclusions; When the parameter c is negative and changing, and λ is complex, all the schemes are asymptotically stable, however as Δt increases and the parameter c is changing, the corresponding $|r|$ values of the schemes decreases accordingly making them more asymptotically stable. When the parameter c is negative and fixed and λ is complex and both are real, all the schemes studied are asymptotically stable, however as Δt increases, corresponding $|r|$ values of the schemes decreases accordingly making them more stable.

- At each Δt, the $|r|$ values of all the schemes are the same, that is; at $\Delta t = 0.001$, $|r|$ value of ETD1 is 0.9999, ETD2 is 0.9999, ETD2RK1 is 0.9999 and ETD2RK2 is 0.9999. Hence as far as asymptotic stability is concern, none of the schemes studied is more stable than the other, therefore ETD1, ETD2, ETD2RK and ETD2RK2 are efficient schemes for solving stiff semi-linear differential equations.
- It will take several Δt values to make the schemes more stable when c is fixed and negative and λ is complex than when c is changing and negative and λ is complex.

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