Persistence Properties and Infinite Propagation Speed for a Two Component B Family System

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Abstract

Considered herein is the behavior of solutions for a two component b family shallow water system. The persistence properties, unique continuation and infinite propagation speed for the solutions of the b family system are examined in this work.

Key words: Two-component b family system; Persistence properties; Propagation speed.

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1. Introduction

This paper is concerned with the evolution of certain solutions to a recently derived two-component b family system, which is given by

$$\begin{cases} m_t+um_x+k_1u_xm+k_2\rho\rho_x=0, & t>0,\ x\in\mathbf{R}, \\ \rho_t+u\rho_x+k_3\rho u_x=0, & t>0,\ x\in\mathbf{R}, \\ m(0,x)=m_0(x), & x\in\mathbf{R}, \\ \rho(0,x)=\rho_0(x), & x\in\mathbf{R}, \end{cases} \tag{1.1}$$

where $m=u-u_{xx}$.

System (1.1) was recently introduced by Guha in [16], this two-component system is defined on a infinite-dimensional Lie group, which is the group of orientation-preserving diffeomorphisms of the circle [30], for more geometric interpretations of system (1.1), see [16, 30] for details.

System (1.1) is a generalization, since it reduces to the following celebrated b family equation upon setting $\rho = 0$:

$$u_t-u_{txx}+(b+1)uu_x=bu_xu_{xx}+uu_{xxx}, \tag{1.2}$$

where b is a constant parameter. Eq.(1.2) can be derived and shown to belong to an asymptotically equivalent family of equations by using Kodamas normal form transformations [25, 26] of the equations that emerge from shallow water asymptotics. In [8, 9], the authors have given a clear explanation of how the CH equation arises from asymptotic expansions for shallow water motion. However, the parameter b may take any value except -1, for which the asymptotic ordering forshallow water is broken. Incidentally the Camassa-Holm equation was recently rederived as a shallow water equation by using asymptotic methods in three diferent approaches by Fokas and Liu in [13], by Dullin et al. in [8, 9] and also by Johnson in [23]. These three derivations used different variants of the method of asymptotic expansions for shallow water waves in the absence of surface tension.

For $b=2$, Eq.(1.2) becomes the Camassa-Holm equation, modelling the unidirectional propagation of shallow water waves over a flat bottom. It is also a model for the propagation of axially symmetric waves in hyperelastic rods [6, 7]. The Cauchy problem of the Camassa-Holm equation has been intensively studied in recent years [3, 4, 5, 27, 31, 35]. On the other hand, persistence properties and unique continuation of solutions to the Camassa-Holm equation has also been derived [21].

For $b=3$, Eq.(1.2) becomes the Degasperis-Procesi equation, it can be regarded as a model for nonlinear shallow water dynamics and its asymptotic accuracy is the same as for the Camassa-Holm equation [8, 9].

A considerable amount of work has been devoted to the study of the corresponding Cauchy problem in both nonperiodic and periodic case, see [11, 28, 32, 33]. Besides, the persistence properties and infinite propagation speed of solution have been investigated in [19] and [20] respectively.

For $\rho \neq 0$, if we take $k_1 = 2$ and $k_3 = 1$, Eq.(1.1) becomes the following two-component system:

$$\begin{cases} m_t + um_x + 2u_x m + k_2 \rho\rho_x = 0, \quad t > 0,\ x \in R, \\ \rho_t + u\rho_x + \rho u_x = 0, \quad t > 0,\ x \in R, \end{cases}$$

where $m = u - u_{xx}$, $k_2 = \pm 1$ was derived by Constantin and Ivanov in the context of shallow water theory. It has a Lax pair and it is bi-Hamiltonian [2]. The mathematical properties of the two-component Camassa-Holm equation have been studied intensively [1, 2, 10, 12, 15, 17, 22, 34]. Among these results, of relevance to the present paper will be the fact that the solutions of the system persist the decaying properties during evolution and have infinite propagation speed.

The aim of this article is to look at how certain solutions of the system (1.1) develop over the duration of their time of existence. First, we will show that the strong solutions of the two-component b family system, initially decaying exponentially together with its spatial derivative, must be identically equal to zero if they also decay exponentially at a later time. Afterwards, we will examine whether the solutions of system (1.1), initially compact supported, will continue to do so as they evolve. We will see that some solutions with remain compactly supported at all future times of their existence, while other solutions display an infinite speed of propagation and instantly lose their compact support.

2. Preliminaries

In this section, we shall address the local well-posedness result of system (1.1). We first introduce some notations. In the following, we denote by $*$ the spatial convolution. Given a Lebesgue space $L^p(\mathrm{R})$, we denote its norm by $\|\cdot\|_p$. Because all spaces of functions are over R, for simplicity, we drop R in our notations of function spaces if there is no ambiguity. In contrast, if there is no particular emphasis, all spatial variable of functions is x, for simplicity, we drop x in our notations of functions if there is no ambiguity. If

$$p(x) = \frac{1}{2}e^{-x}$$

then $(1-\partial_x^2)^{-1} f = p * f$ for all $f \in L^2(\mathrm{R})$ and so $p * m = u$. Applying the convolution operator to the first equation of system (1.1), we can rewrite system (1.1) in the following equivalent form

$$\begin{cases} u_t + uu_x + \partial_x p * (\frac{k_1}{2}u^2 + \frac{3-k_1}{2}u_x^2 + \frac{k_2}{2}\rho^2) = 0, & t>0,\ x \in \mathbf{R}, \\ \rho_t + u\rho_x + k_3 u_x \rho = 0, & t>0,\ x \in \mathbf{R}, \\ u(0,x) = u_0(x), & x \in \mathbf{R}, \\ \rho(0,x) = \rho_0(x), & x \in \mathbf{R}. \end{cases} \tag{2.1}$$

The system (2.1) is the correct form to apply Kato's semigroup theory [24] to show local well-posedness, for details, we refer to [29]. The following result then follows

Theorem 1 Given $X_0 = (u_0, \rho_0)^T \in H^s(\mathbf{R}) \times H^{s-1}(\mathbf{R})$, there exists a maximal existence time $T = T(\|X_0\|_{H^s \times H^{s-1}})$ and an unique solution $X = (u, \rho)^T$ to system (1.1) such that

$$X = X(\cdot, X_0) \in C([0,T); H^s(\mathrm{R}) \times H^{s-1}(\mathrm{R})) \cap C^1([0,T); H^{s-1}(\mathrm{R}) \times H^{s-2}(\mathrm{R})),$$

Moreover, the solution depends continuously on the initial data, i.e. the mapping

$$X_0 \to X(\cdot, X_0) : H^s(\mathrm{R}) \times H^{s-1}(\mathrm{R}) \to C\left([0,T); H^s(\mathrm{R}) \times H^{s-1}(\mathrm{R})\right) \cap C^1([0,T); H^{s-1}(\mathrm{R}) \times H^{s-2}(\mathrm{R}))$$

is continuous.

3. Persistence properties and unique continuation

In this section, we shall investigate the following persistence properties for the strong solution of system (1.1) in L^{∞} space with exponential weights.

Theorem 2 Assume that $X_0 \in H^s(\mathbf{R}) \times H^{s-1}(\mathbf{R})$ with $s \geq 5/2$ satisfies that for some $\theta \in (0, 1)$

$|X_0(x)|, |X_{0x}(x)| \sim O\left(e^{-\theta x}\right)$, as $x \to \infty$,

then the corresponding strong solution to system (1.1) satisfies

$|X(t,x)|, \quad |X_x(t,x)| \sim O(e^{-\theta x})$

uniformly in the time interval [0,T].

Proof For simplicity, we introduce the following notations:

$$F(u,\rho) = \frac{k_1}{2}u^2 + \frac{3-k_1}{2}u_x^2 + \frac{k_2}{2}\rho^2, \quad M = \sup_{t\in[0,T]} \|X(t)\|_{H^s \times H^{s-1}}.$$

Multiplying the first equation in (2.1) by $u^{2p-1}(p \in \mathbf{Z}^+)$ and integrating the result in the x-variable over R, we obtain

$$\int_{\mathbf{R}} u^{2p-1}u_t dx + \int_{\mathbf{R}} u^{2p-1}uu_x dx + \int_{\mathbf{R}} u^{2p-1}\partial_x p * F(u,\rho)dx = 0. \qquad (3.1)$$

In view of the equality

$$\int_{\mathbf{R}} u^{2p-1}u_t dx = \frac{1}{2p}\frac{d}{dt}\|u(t)\|_{2p}^{2p} = \|u(t)\|_{2p}^{2p-1}\frac{d}{dt}\|u(t)\|_{2p},$$

and the estimate

$$\left|\int_{\mathbf{R}} u^{2p-1}uu_x dx\right| \leq \|u_x(t)\|_\infty \|u(t)\|_{2p}^{2p},$$

it can be deduced from (3.1) that

$$\frac{d}{dt}\|u(t)\|_{2p} \leq \|u_x(t)\|_\infty \|u(t)\|_{2p} + \|\partial_x p * F(u,\ \rho)(t)\|_{2p}.$$

The Gronwall's inequality and Sobolev embedding theorem enable us to derive

$$\|u(t)\|_{2p} \leq (\|u(0)\|_{2p} + \int_0^t \|\partial_x p * F(u,\rho)(\tau)\|_{2p} d\tau)e^{\mathrm{Mt}}. \qquad (3.2)$$

Since $f \in L^2 \cap L^\infty$ implies $\lim_{q\to\infty}\|f\|_q = \|f\|_\infty$, and $\partial_x p \in L^1$, $F(u,p) \in L^1 \cap L^\infty$, taking the limits in (3.2), we get

$$\|u(t)\|_\infty \leq (\|u(0)\|_\infty + \int_0^t \|\partial_x p * F(u,\rho)(\tau)\|_\infty d\tau)e^{Mt}.$$

Multiplying the second equation in (2.1) by $\rho^{2p-1}(p \in Z^+)$, we can see

$$\int_{\mathbf{R}} \rho^{2p-1}\rho_t dx + k_3\int_{\mathbf{R}} \rho^{2p-1}\rho u_x dx + \int_{\mathbf{R}} \rho^{2p-1}\rho_x u dx = 0. \qquad (3.3)$$

It follows from the following estimates

$$\int_{\mathbf{R}} \rho^{2p-1}\rho_t dx = \frac{1}{2p}\frac{d}{dt}\|\rho(t)\|_{2p}^{2p-1}\frac{d}{dt}\|\rho(t)\|_{2p} = \|\rho(t)\|_{2p}^{2p-1}\frac{d}{dt}\|\rho(t)\|_{2p},$$

$$\int_{\mathbf{R}} \rho^{2p-1}\rho u_x dx \leq \|u_x\|_\infty \|\rho(t)\|_{2p}^{2p},$$

$$\int_{\mathbf{R}} \rho^{2p-1}\rho_x u dx = \frac{1}{2p}\int_{\mathbf{R}} u(\rho^{2p})_x dx = -\frac{1}{2p}\int_{\mathbf{R}} \rho^{2p}u_x dx \leq \frac{1}{2p}\|u_x\|_\infty \|\rho(t)\|_{2p}^{2p}.$$

and Holder's inequality in (3.3) that

$$\frac{d}{dt}\|\rho(t)\|_{2p} \leq (|k_3|+1)\|u_x\|_\infty \|\rho(t)\|_{2p}.$$

Since for $m > \frac{1}{2}$, the Sobolev embedding theorem $H^m \to L^\infty$ holds, i.e., $\|u_x\|_{L^\infty} \leq \|u\|_{H^s} \leq M$ holds for $s \geq 2$.

The Gronwall's inequality and Sobolev embedding theorem enable us to derive

$$\|\rho(t)\|_{2p} \le \|\rho_0\|_{2p} e^{(|k_3|+1)Mt}.$$

Taking the limit as $p \to \infty$ we get

$$\|\rho(t)\|_{\infty} \le \|\rho_0\|_{\infty} e^{(|k_3|+1)Mt}.$$

Next, differentiating the first equation of system (2.1) with respect to x yields

$$u_{tx} + uu_{xx} + u_x^2 + \partial_x^2 p * F(u,\ \rho) = 0. \qquad (3.4)$$

Multiplying both sides of (3.4) by u_x^{2p-1} $(p \in Z^+)$, an integration by parts yields

$$\int_{\mathbf{R}} uu_{xx}u_x^{2p-1}dx = \frac{1}{2p}\int_{\mathbf{R}} u(u_x^{2p})_x dx = -\frac{1}{2p}\int_{\mathbf{R}} u_x^{2p+1}dx.$$

Therefore,

$$\frac{d}{dt}\|u_x(t)\|_{2p} \le 2\|u_x(t)\|_{\infty}\|u_x(t)\|_{2p} + \left\|\partial_x^2 p * F(u,\ \rho)(t)\right\|_{2p},$$

from which Gronwall's inequality implies

$$\|u_x(t)\|_{2p} \le (\|u_x(0)\|_{2p} + \int_0^t \left\|\partial_x^2 p * F(u,\ \rho)(\tau)\right\|_{2p} d\tau)e^{2Mt}.$$

Since the equality $\partial_x^2 p * f = p * f - f$ holds for all f, taking the limit in the above inequality yields

$$\|u_x(t)\|_{\infty} \le (\|u_x(0)\|_{\infty} + \int_0^t \left\|\partial_x^2 p * F(u,\ \rho)(\tau)\right\|_{\infty} d\tau)e^{2Mt}.$$

In a similar manner, we can derive for the second equation of (2.1) that

$$\frac{d}{dt}\|\rho_x\|_{2p} \le k_3(1+\frac{1}{2p})\|u_x\|_{\infty}\|\rho_x\|_{2p} + k_3\|u_{xx}\|_{\infty}\|\rho\|_{2p},$$

therefore, solving the above diferential inequality, and then taking the limit we get

$$\|\rho_x(t)\|_{\infty} \le e^{Mt}(\|\rho_x(0)\|_{\infty} + \int_0^t \|\rho(\tau)\|_{\infty} d\tau).$$

In order to get the desired result, we introduce the weight function $\psi_n(x)$ which is independent of t as follows

$$\psi_N(x) = \begin{cases} 1, & x \le 0\ , \\ e^{\theta x}, & x \in (0, N), \\ e^{\theta N}, & x \ge N\ . \end{cases}$$

where $N \in Z^+$ and we have $0 \le \psi_N'(x) \le \psi_N(x)$. Multiplying the first equation of (2.1) and (3.4) by $\psi_N(x)$, one can derive

$$\partial_t(u\psi_N) + (u\psi_N)u_x + \psi_N\partial_x p * F(u, \rho) = 0, \qquad (3.5)$$

$$\partial_t(u_x\psi_N) + (u_x\psi_N)u_x + uu_{xx}\psi_N + \psi_N\partial_x^2 p * F(u, \rho) = 0, \qquad (3.6)$$

Again, multiplying (3.5) by $(u\psi_N)^{2p-1}$ and (3.6) by $(u_x\psi_N)^{2p-1}$ respectively, then integrating the resulting equality over R, we have by integrating by parts

$$\left|\int_{\mathbf{R}} uu_{xx}\psi_N(u_x\psi_N)^{2p-1}dx\right| = \left|\int_{\mathbf{R}} u(u_x\psi_N)^{2p-1}\left[\partial_x(u_x\psi_N) - u_x\psi_N'\right]dx\right|$$

$$= \left|\frac{1}{2p}\int_{\mathbf{R}} u\partial_x\left[(u_x\psi_N)^{2p}\right]dx - \int_{\mathbf{R}} uu_x\psi_N'(u_x\psi_N)^{2p-1}dx\right|$$

$$= \left|\frac{1}{2p}\int_{\mathbf{R}} u_x(u_x\psi_N)^{2p}dx + \int_{\mathbf{R}} uu_x\psi_N'(u_x\psi_N)^{2p-1}dx\right| \le 2(\|u(t)\|_{\infty} + \|u_x(t)\|_{\infty})\|u_x\psi_N\|_{2p}^{2p}.$$

Therefore, similar as the weightless case, we get

$$\|u\psi_N\|_\infty + \|u_x\psi_N\|_\infty \le e^{2Mt}(\|u_0\psi_N\|_\infty + \|u_x(0)\psi_N\|_\infty)$$
$$+e^{2Mt}\int_0^t (\|\psi_N\partial_x p * F(u,\rho)\|_\infty + \|\psi_N\partial_x^2 p * F(u,\rho)\|_\infty)d\tau. \qquad (3.7)$$

Multiplying the second equation of (2.1) and its diferential form with respect to by ψ_N, we have

$$(\rho\psi_N)_t + k_3u_x(\rho\psi_N) + \rho_x u\psi_N = 0, \qquad (3.8)$$
$$\partial_t(\rho_t\psi_N) + u\rho_{xx}\psi_N + (k_3+1)u_x(\rho_x\psi_N) + k_3u_{xx}\rho\psi_N = 0. \qquad (3.9)$$

Multiplying (3.8) and (3.9) with $(\rho\psi_N)^{2p-1}$ and $(\rho_x\psi_N)^{2p-1}$ respectively, an integration by parts yields for (3.8)

$$\int_{\mathbf{R}}(\rho\psi_N)^{2p-1}(\rho\psi_N)_t dx = \|\rho\psi_N\|_{2p}^{2p-1}\frac{d}{dt}\|\rho\psi_N\|_{2p},$$
$$\int_{\mathbf{R}}\rho_x u\psi_N(\rho\psi_N)^{2p-1}dx = \int_{\mathbf{R}}u(\rho\psi_N)^{2p-1}\left[\partial_x(\rho\psi_N) - \rho\psi_N'\right]dx$$
$$= \frac{1}{2p}\int_{\mathbf{R}}u\partial_x(\rho\psi_N)^{2p}dx - \int_{\mathbf{R}}u\rho\psi_N'(\rho\psi_N)^{2p-1}dx = -\frac{1}{2p}\int_{\mathbf{R}}u_x(\rho\psi_N)^{2p}dx - \int_{\mathbf{R}}u\rho\psi_N'(\rho\psi_N)^{2p-1}dx.$$

The estimates for (3.9) can be derived in a similar manner. Thus we can obtain

$$(\|\rho\psi_N\|_\infty + \|\rho_x\psi_N\|_\infty) \le e^{2Mt}(\|\psi_N\rho(0)\|_\infty + \|\psi_N\rho_x(0)\|_\infty). \qquad (3.10)$$

On the other hand, simple calculation shows that there exists $C>0$ only depending on $\theta\in(0,1)$ such that

$$\psi_N(x)\int_{\mathbf{R}}e^{-|x-y|}\frac{1}{\psi_N(y)}dy \le C.$$

Therefore, one gets for $F(u,\rho)$ and a general constant $C>0$, such that

$$|\psi_N\partial_x p * F(u,\rho)| = \left|\frac{1}{2}\psi_N(x)\int_{\mathbf{R}}\mathrm{sgn}(x-y)e^{-|x-y|}\left(\frac{k_1}{2}u^2 + \frac{3-k_1}{2}u_x^2 + \frac{k_2}{2}\rho^2\right)dy\right|$$
$$\le \frac{1}{2}\psi_N(x)\left|\int_{\mathbf{R}}e^{-|x-y|}\frac{1}{\psi_N(y)}\psi_N(y)\left(\frac{k_1}{2}u^2 + \frac{3-k_1}{2}u_x^2 + \frac{k_2}{2}\rho^2\right)dy\right|$$
$$\le C(\|\psi_N u\|_\infty\|u\|_\infty + \|\psi_N u_x\|_\infty\|u_x\|_\infty + \|\rho\psi_N\|_\infty\|\rho\|_\infty).$$

By using the relation $\partial_x^2 p * f = p * f - f$ again one can derive

$$|\psi_N\partial_x^2 p * F(u,\rho)| = \left|\frac{1}{2}\psi_N(x)\int_{\mathbf{R}}e^{-|x-y|}F(u,\rho)dy - \psi_N(x)\left(\frac{k_1}{2}u^2 + \frac{3-k_1}{2}u_x^2 + \frac{k_2}{2}\rho^2\right)(x)\right|$$
$$\le \frac{1}{2}\psi_N(x)\left|\int_{\mathbf{R}}e^{-|x-y|}\frac{1}{\psi_N(y)}\psi_N(y)\left(\frac{k_1}{2}u^2 + \frac{3-k_1}{2}u_x^2 + \frac{k_2}{2}\rho^2\right)dy\right| + \left|\psi_N(x)\left(\frac{k_1}{2}u^2 + \frac{3-k_1}{2}u_x^2 + \frac{k_2}{2}\rho^2\right)(x)\right|$$
$$\le C(\|\psi_N u\|_\infty\|u\|_\infty + \|\psi_N u_x\|_\infty\|u_x\|_\infty + \|\rho\psi_N\|_\infty\|\rho\|_\infty).$$

Substituting all the above estimates into (3.7), we can deduce from (3.10) that there exists a positive constant C such that

$$\|\psi_N u\|_\infty + \|\psi_N u_x\|_\infty + \|\rho\psi_N\|_\infty + \|\rho_x\psi_N\|_\infty$$
$$\le C(\|\psi_N u_0\|_\infty + \|\psi_N u_x(0)\|_\infty + \|\psi_N\rho_0\|_\infty + \|\psi_N\rho_x(0)\|_\infty)$$
$$+C\int_0^t(\|u\|_\infty + \|u_x\|_\infty + \|\rho\|_\infty + \|\rho_x\|_\infty)(\|\psi_N u\|_\infty + \|\psi_N u_x\|_\infty + \|\psi_N\rho\|_\infty + \|\psi_N\rho_x\|_\infty)dt,$$

Therefore, the application of Gronwall's inequality in integral form yields for given $N\in ¢$ and $t\in[0,T)$

$$\|\psi_N u\|_\infty + \|\psi_N u_x\|_\infty + \|\rho\psi_N\|_\infty + + \|\rho_x\psi_N\|_\infty$$
$$\le C(\|\psi_N u_0\|_\infty + \|\psi_N u_x(0)\|_\infty + \|\psi_N\rho_0\|_\infty + \|\psi_N\rho_x(0)\|_\infty)$$
$$\le C(\|e^{\theta x}u_0\|_\infty + \|e^{\theta x}u_x(0)\|_\infty + \|e^{\theta x}\rho_0\|_\infty + \|e^{\theta x}\rho_x(0)\|_\infty).$$

By taking $N \to \infty$ in the above estimate, we can see

$$\sup_{t\in[0,T]} (\|e^{\theta x}u\|_\infty + \|e^{\theta x}u_x\|_\infty + \|e^{\theta x}\rho\|_\infty + \|e^{\theta x}\rho_x\|_\infty)$$
$$\le C(\|e^{\theta x}u_0\|_\infty + \|e^{\theta x}u_x(0)\|_\infty + \|e^{\theta x}\rho_0\|_\infty + \|e^{\theta x}\rho_x(0)\|_\infty).$$

Thus

$$|u(t,x)|,\ \ |\partial_x u(t,x)|,\ \ |\rho(t,x)|,\ \ |\partial_x \rho(t,x)| \sim O\left(e^{-\theta x}\right), \text{ as } x \to \infty$$

uniformly in the time interval $[0,T)$, Theorem 2 is so proved.

Analogous to the proof of Theorem 2, we can conclude the following

Corollary 1 Assume that $X_0 \in H^s(\mathbf{R}) \times H^{s-1}(\mathbf{R})$ with $s \ge 5/2$ satisfies that for some $\theta \in (0,1)$

$$|u_0(x)|,\ \ |\partial_x u_0(x)|,\ \ |\rho_0(x)| :\ O(e^{-\theta x}),$$

then the corresponding strong solution to system (1.1) satisfies

$$|u(t,x)|,\ \ |\partial_x u(t,x)|,\ \ |\rho(t,x)| :\ O(e^{-\theta x}).$$

Uniformly on the time interval $[0,T)$.

The following theorem which concerns on the unique continuation property is based on the above persistence property, it is to formulate decay conditions on a solution, at two distinct times, in two distinct cases, which guarantee that $X=(0,0)^T$ and $X=(0,\rho)^T$ is the unique solution of the following two cases respectively.

Theorem 3 Assume that $X_0 \in H^s(\mathbf{R}) \times H^{s-1}(\mathbf{R})$ with $s \ge 2$ and $0 < k_1 \le 3$,If for some $\theta \in (\frac{1}{2},1)$,

(1) For $k_2 > 0$, $|u_0(x)| \sim o(e^{-x})$, $|\partial_x u_0(x)| \sim O(e^{-\theta x})$, $|\rho_0(x)| :\ O(e^{-\theta x})$, as $x \to \infty$,

and there exists $t_1 \in (0,T]$ such that $|u(t_1,x)| \sim o(e^{-x})$ as $x \to \infty$. Then $u \equiv 0, \rho \equiv 0$.

(2) For $k_2 = 0$, $|u_0(x)| \sim o(e^{-x})$, $|\partial_x u_0(x)| \sim O(e^{-\theta x})$, as $x \to \infty$, and there exists $t_1 \in (0,T]$ such that $|u(t_1,x)| \sim o(e^{-x})$ as $x \to \infty$. Then $u \equiv 0, \rho(t,x) \equiv \rho_0(x)$.

Proof (1) Integrating the first equation of (2.1) with respect to time t over $[0, t_1]$ yields

$$u(x,t_1) - u(x,0) + \int_0^{t_1} uu_x(x,\tau)d\tau = -\int_0^{t_1} \partial_x p * (\frac{k_1}{2}u^2 + \frac{3-k_1}{2}u_x^2 + \frac{k_2}{2}\rho^2)d\tau. \qquad (3.11)$$

In view of the assumption of the theorem, we can derive

$$u(x,t_1) - u(x,0) \sim o(e^{-x}), \text{ as } x \to \infty.$$

It follows from Corollary 1 that $\int_0^{t_1} uu_x(x,t)dx \sim O(e^{-2\theta x})$, Therefore, $\int_0^{t_1} uu_x(x,t)dx \sim o(e^{-x})$.

For the right hand side of (3.11), we have

$$\int_0^{t_1} \partial_x p *(\frac{k_1}{2}u^2 + \frac{3-k_1}{2}u_x^2 + \frac{k_2}{2}\rho^2)dt = \partial_x p * \int_0^{t_1} (\frac{k_1}{2}u^2 + \frac{3-k_1}{2}u_x^2 + \frac{k_2}{2}\rho^2)d\tau := \partial_x p * Q(x).$$

Since $k_2 > 0$, it follows from Corollary 1 again $0 \le Q(x) \sim O(e^{-2\theta x})$, thus we have $Q(x) \sim o(e^{-x})$ as $x \to \infty$. In consequence

$$\partial_x p * Q(x) = -\frac{1}{2}\int_R \operatorname{sgn}(x-y)e^{-|x-y|}Q(y)dy = -\frac{1}{2}e^{-x}\int_{-\infty}^x e^y Q(y)dy + \frac{1}{2}e^x \int_x^{+\infty} e^{-y}Q(y)dy,$$

while $e^x \int_x^{+\infty} e^{-y}Q(y)dy = o(1)e^x \int_x^{+\infty} e^{-2y}dy = o(1)e^{-x} \sim o(e^{-x})$.

If $Q \ne 0$, then there exists $C_0 > 0$, such that for $x ? \ 1$, $\int_{-\infty}^x e^y Q(y)dy \ge C_0$. Therefore, we have for $x ? \ 1$ that $-\partial_x p * Q(x) \ge \frac{C_0}{2}e^{-x}$, from which we obtain a contradiction, thus $Q(x) \equiv 0$, which implies $u \equiv 0, \rho \equiv 0$.

(2) For $k_2 = 0$, using the discussion above we can obtain $u = 0$, substituting into the second equation of (2.1) we can deduce $\rho_t = 0$, it is easy to observe by now that $\rho(t,x) = \rho_0(x)$.
Analogous to Theorem 3, we can derive the following

Theorem 4 Assume that $X_0 \in H^s(\mathbf{R}) \times H^{s-1}(\mathbf{R})$ with $s \geq 2$**,** if for some $\theta \in (\frac{1}{2}, 1)$

$$|u_0(x)| \sim O(e^{-x}), \quad |\partial_x u_0(x)| \sim O(e^{-\theta x}), \quad |\rho_0(x)| : \ O(e^{-\theta x}), \text{ as } x \to \infty,$$

then $|u(t,x)| \sim O(e^{-x}),\ \rho(t,x) \sim O(e^{-\theta x})$ uniformly in time interval $[0,T]$.

4. Infinite propagation speed

Recently, Guo and Ni [18] proved the infinite propagation speed for a two component generalized Camassa-Holm equation by establishing a detailed description on the profile of the corresponding solution with compactly supported initial datum. But how about the generalized two component family system? Do this kind of system have the same infinite propagation speed as the two component Camassa-Holm equation, the answer is positive.
In the following we make use of the family $\{q(\tau,\cdot)\}_{\tau \in [0,T]}$ of difeomorphism defined by

$$\begin{cases} q_t(t,x) = u(t, q(t,x)), \\ q(x,0) = x. \end{cases} \tag{4.1}$$

For the Camassa-Holm equation, these diffeomorphisms have a geometric interpretation, however, there is no such interpretation for this two component *b* family system yet. By solving (4.1), one easily obtains

$$q_x(t,x) = e^{\int_0^t u_x(s, q(s,x)) ds}, \tag{4.2}$$

which tells us these diffeomorphisms $q(t,\cdot)$ are increasing for each $t \in [0,T)$.

The following Lemma tells us that the solution $\rho(t,x)$ shares the same sign with its initial value as it evolves, moreover, if $\rho_0(x)$ is compactly supported, so do the solution $\rho(t,x)$.

Lemma 1 Assume that $X_0 \in H^s(\mathbf{R}) \times H^{s-1}(\mathbf{R})$ with $s \geq 2$, T is the maximal existence time of solution to system $X = (u,\rho)^T$, then for all $(t,x) \in [0,T) \times ¡$,

$$\rho(t,q) q_x^{k_3}(t,x) = \rho_0(x). \tag{4.3}$$

Proof of Lemma 1 Differentiating (4.2) with respect totime variable, in view of the second equation of (2.1), we have

$$\begin{aligned} \frac{d}{dt}\left\{\rho(t,q) q_x^{k_3}(t,x)\right\} &= (\rho_t(t,q) + \rho_x(t,q) q_t(t,x)) q_x^{k_3}(t,x) + k_3 \rho(t,q) q_x^{k_3-1}(t,x) q_{tx}(t,x) \\ &= \left[\rho_t(t,q) + \rho_x(t,q) q_t(t,x) + k_3 \rho(t,q) u_x(t,q)\right] q_x^{k_3}(t,x) \\ &= 0. \end{aligned}$$

Lemma 2 Assume that u_0 is such that $m_0 = u_0 - u_{0,xx}$ has compact support, contained in the interval $[a_0, b_0]$, and ρ_0 is also compactly supported with its support contained in $[a_1, b_1]$. If $T > 0$ is the maximal existence time of the unique classical solution $X = (u,\rho)$ to system (1.1) with the given initial data $X_0 = (u_0, \rho_0)$, then for any $t \in [0,T)$ the solution $m(t,x)$ has compact support.

Proof It follows from the first equation of (1.1) and the diffeomorphism (4.1) that

$$\begin{aligned} \frac{d}{dt}\left[m(t, q(t,x) q_x^{k_1}(t,x)\right] &= (m_t + m_x q_t) q_x^{k_1}(t,x) + k_1 m q_x^{k_1-1} q_{tx}(t,x) \\ &= (m_t + m_x q_t) q_x^{k_1} + k_1 m u_x q_x^{k_1} = -k_2 \rho \rho_x q_x^{k_1}, \end{aligned}$$

therefore,

$$m(t_1, q(t,x))q_x^{k_1} = m_0(x) - \int_0^t k_2 \rho(q(s,x),\ s)\rho_x(q(s,x),\ s)q_x^{k_1}(s,x)ds. \qquad (4.4)$$

Note that (4.3) and the assumption of the lemma guarantee $\rho(t,x)$ has compact support in $[q(a_1,t), q(b_1,t)]$, and the integral part of (4.4) is also compactly supported in $[a_1, b_1]$. Thus, if we

Take $a = \max\{a_0, a_1\}, b = \min\{b_0, b_1\}$, we can see $m(t,\cdot)$ is compactly supported in $[a_1, b_1]$.

If we suppose has compact support, then clearly m_0 is also compactly supported, and by Lemma 2 the function m will then have compact support for all $t \in [0,T)$. But this property is not valid for the solution component u, actually, in the following we shall prove that the only solution of (1.1) which retains the property of having compact support for any further time is the trivial solution $u \equiv 0$. The solution u has an infinite propagation speed in the sense that it instantly loses its compact support.

First, we introduce a lemma

Lemma 3 [14] Let $u \in C^2(\mathbf{R}) \cap H^2(\mathbf{R})$ be such that $m = u - u_{xx}$ has compact support. Then u has compact support if and only if

$$\int_{\mathbf{R}} e^x m(x)dx = \int_{\mathbf{R}} e^{-x} m(x)dx = 0. \qquad (4.5)$$

Theorem 5 Assume that the initial data u_0 has compact support, $T > 0$ be the maximal existence time of the unique solution $u(t,x)$ with initial data u_0, $0 < k_1 \le 3$, $k_2 > 0$. If at every point $t \in [0,T)$ the C^2 function $u(t,x)$ has compact support, then $u = \rho \equiv 0$.

Proof Differentiating the first integral of (4.5) with respect to t with the first equation in (1.1) applied, we get

$$\begin{aligned}
\frac{d}{dt}\int_{\mathbf{R}} e^x m(t,\ x)dx &= \int_{\mathbf{R}} e^x m_t dx \\
&= -k_1\int_{\mathbf{R}} e^x m u_x dx - \int_{\mathbf{R}} e^x m_x u dx - k_2\int_{\mathbf{R}} e^x \rho\rho_x dx \\
&= -k_1\int_{\mathbf{R}} e^x m u_x dx + \int_{\mathbf{R}} e^x m u_x dx + \int_{\mathbf{R}} e^x m u dx + \frac{k_2}{2}\int_{\mathbf{R}} e^x \rho^2 dx \\
&= (1-k_1)\int_{\mathbf{R}} e^x m u_x dx + \int_{\mathbf{R}} e^x m u dx + \frac{k_2}{2}\int_{\mathbf{R}} e^x \rho^2 dx \\
&= (1-k_1)\int_{\mathbf{R}} e^x u u_x dx - (1-k_1)\int_{\mathbf{R}} e^x u_{xx} u_x dx + \int_{\mathbf{R}} e^x u^2 dx - \int_{\mathbf{R}} e^x u_{xx} u dx + \frac{k_2}{2}\int_{\mathbf{R}} e^x \rho^2 dx \\
&= \frac{k_1}{2}\int_{\mathbf{R}} e^x u^2 dx + \frac{3-k_1}{2}\int_{\mathbf{R}} e^x u_x^2 dx + \frac{k_2}{2}\int_{\mathbf{R}} e^x \rho^2 dx,
\end{aligned}$$

Therefore, we can derive

$$\frac{d}{dt}\int_{\mathbf{R}} e^x m(t,\ x)dx = \int_{\mathbf{R}} e^x \left(\frac{k_1}{2}u^2 + \frac{3\text{-}k_1}{2}u_x^2 + \frac{k_2}{2}\rho^2\right)dx. \qquad (4.6)$$

It follows from the assumption of the theorem and Lemma 4.3 that $u(t,x)$ is compactly supported, Thus $u = \rho \equiv 0$.

The important work in this section is to give a more detailed description on the corresponding strong solution $X(t,x)$ in its life span with X_0 being compactly supported. The main theorem reads

Theorem 6 Assume that for some $T>0$ and $s\geq 2$, $X\in C([0,T);H^s(\mathrm{R})\times H^{s-1}(\mathrm{R}))$ is the unique strong solution of system (1.1), $0<k_1\leq 3$, $k_2>0$. If the initial data $u_0(x)=u(0,x)$ is compactly supported in $[\alpha_0,\beta_0]$ and the initial data $u_0(x)=u(0,x)$ is compactly supported in $[\alpha_1,\beta_1]$, then we have for any $t\in(0,T)$,

$$u(t,x)=\begin{cases}E_+(t)e^{-x}, & x>q(\beta,t),\\ E_-(t)e^{x}, & x<q(\alpha,t).\end{cases} \tag{4.7}$$

Where $\alpha=\max\{\alpha_0,\alpha_1\}$, $\beta=\min\{\beta_0,\beta_1\}$, $\eta_+(t)$, $\eta_-(t)$ are continuous functions satisfying $\eta_+(0)=\eta_-(0)=0$ for all $t\in(0,T)$, with $\eta_+(t)$ being a strictly increasing function, while $\eta_-(t)$ being strictly decreasing.

Proof By using the relation $u=p*m$ we can decompose u as following

$$u(x)=\frac{1}{2}e^{-x}\int_{-\infty}^{x}e^{y}m(y)dy+\frac{1}{2}e^{x}\int_{x}^{+\infty}e^{-y}m(y)dy.$$

We define functions

$$\eta_+(t)=\int_{q(\alpha,t)}^{q(\beta,t)}e^{y}m(t,y)dy,\quad \eta_-(t)=\int_{q(\alpha,t)}^{q(\beta,t)}e^{-y}m(t,y)dy,$$

Therefore we have

$$u(t,x)=\frac{1}{2}e^{-x}\eta_+(t),\qquad x>q(\beta,t).$$

$$u(t,x)=\frac{1}{2}e^{x}\eta_-(t),\qquad x<q(\alpha,t).$$

Differentiating the above two equalities with respect to x yields

$$\frac{1}{2}e^{-x}\eta_+(t)=u(t,x)=-u_x(t,x)=u_{xx}(t,x),\quad x>q(\beta,t),$$

$$\frac{1}{2}e^{x}\eta_-(t)=u(t,x)=u_x(t,x)=u_{xx}(t,x),\quad x<q(\alpha,t).$$

Since $u(0,\cdot)$ is compactly supported in $[\alpha,\beta]$, we can deduce $\eta_+(0)=\eta_-(0)=0$.

For each fixed t, since $m(t,\cdot)$ is compactly supported in $[q(t,\alpha),q(t,\beta)]$, we can derive

$$\frac{d\eta_+(t)}{dt}=\int_{q(\alpha,t)}^{q(\beta,t)}e^{y}m_t(t,y)dy=\int_{-\infty}^{+\infty}e^{y}m_t(t,y)dy.$$

In view of (4.6), we have

$$\frac{d\eta_+(t)}{dt}=\int_{-\infty}^{+\infty}e^{y}\left(\frac{k_1}{2}u^2+\frac{3-k_1}{2}u_x^2+\frac{k_2}{2}\rho^2\right)dy,\quad t\in[0,T).$$

From which we can obtain $\dfrac{d\eta_+(t)}{dt}>0,\ t\in[0,T)$.

A similar calculation yields

$$\frac{d\eta_-(t)}{dt}=\int_{q(\alpha,t)}^{q(\beta,t)}e^{-y}m_t(t,y)dy=\int_{-\infty}^{+\infty}e^{-y}m_t(t,y)dy.$$

Integrating by parts enable us to derive

$$\frac{d\eta_-(t)}{dt}=-\int_{-\infty}^{+\infty}e^{-y}m_t(t,y)dy=-\int_{-\infty}^{+\infty}e^{-y}\left(\frac{k_1}{2}u^2+\frac{3-k_1}{2}u_x^2+\frac{k_2}{2}\rho^2\right)dy,\ t\in[0,T).$$

It is easy to observe from above that $\dfrac{d\eta_-(t)}{dt}<0,\ t\in[0,T)$.

Therefore, in the lifespan of the solution, we have that $\eta_+(t)$ is an increasing function with $\eta_+(0)=0$, thus it follows that for $t \in [0, T)$, $\eta_+(t)$ can be expressed as

$$\eta_+(t) = \int_0^t \int_R e^x (\frac{k_1}{2}u^2 + \frac{3-k_1}{2}u_x^2 + \frac{k_2}{2}\rho^2) dx d\tau > 0.$$

And similarly, $\eta_-(t)$ can be expressed as

$$\eta_+(t) = -\int_0^t \int_R e^x (\frac{k_1}{2}u^2 + \frac{3-k_1}{2}u_x^2 + \frac{k_2}{2}\rho^2) dx d\tau < 0.$$

In order to finish the proof, it is sufficient to let $E_+(t) = \frac{1}{2}\eta_+(t)$ and $E_-(t) = \frac{1}{2}\eta_-(t)$ respectively.

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