On Turbulent Boundary Layer Development along a Smooth Flat Plate at Zero Incidence and Zero Pressure Gradient

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Abstract

Incorporating the one-seventh power law in the integral form of momentum equation in conjunction with the expression for the turbulent viscous shear stress, we establish relations for the turbulent boundary layer thickness as well as for viscous shear stress. The analysis is based on a steady, turbulent, constant property, two dimensional boundary layer flow over a flat plate at zero angle of approach and zero pressure gradient. We proceed to obtain a shape factor typical of the turbulent flow. Comparison of the exact and approximate values of the boundary layer thickness and shape factor leads to the determination of the percentage error for each of these characteristics. It is observed that as the boundary layer flow becomes more turbulent (i.e Re >> 5×10^5), where R_e is the Reynolds number, these parameters decrease in value with consequent increase in the percentage error.

Keywords: Turbulent, boundary layer, flow, smooth, flat plate, zero incidence, zero pressure gradient, two dimensional.

1.0 Introduction

A boundary layer is the layer of fluid in the immediate vicinity of a boundary surface where the effects of viscosity are significant. The turbulent region of the boundary layer consists of three layers, viz: the laminar sublayer, the buffer layer and the turbulent layer. The laminar sublayer is a thin region next to the wall in which the flow is laminar. It is separated from the turbulent layer by the buffer layer which is a region of transition from laminar to turbulent flow. The buffer layer is so thin that it is always ignored in boundary layer analysis. In both regions of the boundary layer, the velocity increases from zero at the plate and attains the free stream velocity U at the outer edge of the boundary layer. The outer edge is usually taken as the point where $u=0.99U$ (Prandtl, [9]), u is the velocity component in the x-direction.

The subject, boundary layer, has been discussed extensively by many authors since the development of the concept by Prandtl [9]. For instance Craft and Lowell [3] applied steady state boundary layer theory to two aspects of oceanic hydrothermal heat flux and in their analysis they showed that, for near-axis model, heat transfer in the hydrothermal boundary layer is greater than the input from steady state generation of the oceanic crust by sea flow spreading.

Afzal [1] used asymptotic arguments to analyse a turbulent boundary layer subjected to a strong adverse pressure gradient. He found that there is an inertial sublayer where the streamwise velocity distribution obeys a half power law, whose slope depends on a non-dimensional parameter $\wedge = \tau_w / p_x \delta$, where τ_w is the wall shear stress, p_x is the pressure gradient and δ is the boundary layer thickness.

Habib et al [5] carried out transient calculation of the boundary layer flow over spills using simulation and experimental approaches. They validated their results against experimental data and also made comparison of the simulated results with empirical prediction models.

Dorfman [4] presented a review of universal functions widely used in different areas of boundary layer theory for many years up to the present. In his work he adopted various solutions from many published articles to show the breadth of universal approaches with application in laminar, turbulent and transition boundary layers in solving non-isothermal and conjugate heat transfer problems as well as in planetary boundary layer problems in meteorology.

Other researchers in the subject worth mentioning include Blasius [2], Kim and Changhoon [6], Mahmoudian and Scales [7], Pohlhausen [8], Schlichting [10], Vyas and Ranjan [11]

In this work we employ a one-seventh power law in integral momentum equation in conjunction with Blasius law of the shear stress to determine the approximate value of the turbulent boundary layer thickness. The analysis leads to the determination of the percentage error by comparing our approximate value with the exact Blasius value of the boundary layer thickness. The work is also analysed graphically.

2.0 Boundary Layer Equations

In the case of steady, two dimensional flow at zero incidence and zero pressure gradient the boundary layer equations due to Blasius [2] are:

Continuity Equation:
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{2.1}$$

Momentum Equation:
$$u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial y} = \upsilon \frac{\partial^2 u}{\partial y^2} \tag{2.2}$$

(Navier-Stokes Equation)

B.Cs.:
$$\left.\begin{array}{l} u = v = 0 \quad at \quad y = 0 \\ u = U \quad at \quad y = \infty \end{array}\right\} \tag{2.3}$$

Where *u*, *v* are the velocity components in the *x,y* directions respectively, while *U* is the free stream velocity.

2.1 Blasius Typical Values for Some Turbulent Characteristics

These include:

(i) Blasius law (or relation) for viscous shear stress near the plate (see Blasius [2]) is given by

$$\tau_{Blasius} = 0.0226\, \rho U^2 \left(\frac{\mu}{\rho U \delta}\right)^{\frac{1}{4}} \tag{2.4}$$

In terms of local skin friction coefficient C_f, (2.4) can be expressed as

(ii)
$$C_f = \frac{\tau_{Blasius}}{\frac{1}{2}\rho U^2} = 0.0452 \left(\frac{\upsilon}{U\delta}\right)^{\frac{1}{4}} \tag{2.5}$$

Where $\tau_{Blasius} =$ Blasius law of wall shear stress, $\upsilon = \dfrac{\mu}{\rho} =$ kinematic viscosity, while μ and φ are the dynamic viscosity and fluid density respectively.

(iii) **Boundary Layer Thickness δ**

$$\frac{\delta_{Blasius}}{x} = \frac{0.382}{(\mathrm{Re}_x)^{\frac{1}{5}}} \tag{2.6}$$

(iv) **Shape Factor H***

$$H^* = 1.3 - 1.4 \tag{2.7}$$

2.2 Derivation of Integral Momentum Equation from (1), (2) and (3).

From the continuity equation (2.1) we find

$$\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x} \tag{2.8}$$

Integrating this wrt *y* we get

$$\int_0^u \frac{\partial v}{\partial y} dy = -\int_0^u \frac{\partial u}{\partial x} dy$$

Or

$$v = -\int_0^u \frac{\partial u}{\partial x} dy \qquad (2.9)$$

Substituting (2.9) into (2.2) we have

$$\frac{u\partial u}{\partial x} - \frac{\partial u}{\partial y}\int_0^u \frac{\partial u}{\partial x} dy = \upsilon \frac{\partial^2 u}{\partial y^2} \qquad (2.10)$$

Integrating (2.10) over *y* from *y=0* to δ, gives

$$\int_0^\delta \left[u\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}\int_0^y \frac{\partial u}{\partial x} dy \right] = \upsilon \int_0^\delta \frac{\partial u}{\partial y^2} dy \qquad (2.11)$$

Integrating the second term of (2.11) by parts we find

$$\int_0^\delta \left[\frac{\partial u}{\partial y}\int_0^y \frac{\partial u}{\partial x} dy \right] dy = u\int_0^y \frac{\partial u}{\partial x} dy \ \Big|_{y=0}^{y=\delta} - \int_0^\delta u\frac{\partial u}{\partial x} dy$$

Or

$$\int_0^\delta \left[\frac{\partial u}{\partial y}\int_0^y \frac{\partial u}{\partial x} dy \right] dy = U_0\int_0^\delta \frac{\partial u}{\partial x} dy - \int_0^\delta u\frac{\partial u}{\partial x} dy \qquad (2.12)$$

Substituting (2.12) into (2.11) yields

$$\int_0^\delta u\frac{\partial u}{\partial y} dy - U_0\int_0^\delta \frac{\partial u}{\partial x} dy + \int_0^\delta \frac{\partial u}{\partial x} dy = \upsilon \int_0^\delta \frac{\partial^2 u}{\partial y^2} dy \qquad (2.13)$$

Carrying out the required integration in (2.13) leads (after evaluation) to

$$2\int_0^\delta u\frac{\partial u}{\partial x} dy - U_0\int_0^\delta \frac{\partial u}{\partial x} dy = \upsilon \frac{\partial u}{\partial y} \Big|_{y=0}^{y=\delta} = \upsilon \frac{\partial u}{\partial y} \Big|_{y=\delta} - \upsilon \frac{\partial u}{\partial y} \Big|_{y=0}$$

Since $\frac{\partial u}{\partial y} \Big|_{y=\delta} = 0$ and noting that the integration is carried out with respect to *y*, we take the differentiation with respect to *x* outside the integral to get

$$\frac{\partial}{\partial x}\int_0^\delta U_0 u dy - \frac{\partial}{\partial x}\int_0^\delta u^2 dy = \upsilon \left(\frac{\partial u}{\partial y} \right)_{y=0} \qquad (2.14)$$

Since the flow is parallel to x-axis, and $\mu \left(\frac{\partial u}{\partial y} \right)_{y=0} = \tau_0$, we can rewrite (2.14) as

$$\frac{d}{dx}\int_0^\delta \left(U_0 u - u^2 \right) dy = \frac{\tau_0}{\rho}$$

or

$$\frac{d}{dx}\int_0^\delta \left(U_0 - u \right) dy = \frac{\tau_0}{\rho}, \qquad (2.15)$$

Equation (2.15) is the integral momentum equation for a steady, planar, two-dimensional boundary layer-type flow with zero pressure gradient, where τ_0 = local shear stress.

3.0 Application of Velocity Distribution in Karman Pohlhausen Equation

The velocity profile typical of turbulent boundary layer flow follows a one-seventh power law of the form

$$\frac{u}{U} = \left(\frac{y}{\delta}\right)^{\frac{1}{7}} \tag{3.1}$$

for Reynolds number *Re* satisfying $5\times10^5 < \mathrm{Re} < 10^7$.

Substituting (3.1) and (2.4) into (2.15) we find

$$\frac{d}{dx}\int_0^\delta U\left(\frac{y}{\delta}\right)^{\frac{1}{7}}\left[u\left(\frac{y}{\delta}\right)^{\frac{1}{7}} - U\left(\frac{y}{\delta}\right)^{\frac{1}{7}}\right]dy = \frac{1}{\rho}\times 0.0226\,\rho U^2\left[\frac{\mu}{\rho U\delta}\right]^{\frac{1}{4}} \tag{3.2}$$

i.e

$$\frac{d}{dx}\int_0^\delta \left[U^2\left(\frac{y}{\delta}\right)^{\frac{1}{7}} - U^2\left(\frac{y}{\delta}\right)^{\frac{2}{7}}\right]dy = 0.0226\,U^2\left[\frac{\mu}{\rho U\delta}\right]^{\frac{1}{4}} \tag{3.3}$$

i.e

$$\frac{d}{dx}\int_0^\delta \left[\left(\frac{y}{\delta}\right)^{\frac{1}{7}} - \left(\frac{y}{\delta}\right)^{\frac{2}{7}}\right]dy = 0.0226\left[\frac{\mu}{\rho U}\right]^{\frac{1}{4}}\delta^{-\frac{1}{4}} \tag{3.4}$$

Integrating the lhs of (3.4) and equating with the rhs, we find

$$\frac{d}{dx}\left\{\left[\frac{7}{8}y^{\frac{8}{7}}\delta^{-\frac{1}{7}} - \frac{7}{9}y^{\frac{9}{7}}\delta^{-\frac{2}{7}}\right]\Big|_{y=0}^{y=\delta}\right\} = 0.0226\left[\frac{\mu}{\rho U}\right]^{\frac{1}{4}}\delta^{-\frac{1}{4}} \tag{3.5}$$

Simplification of (3.5) yields

$$\frac{7}{72}\;\frac{d}{dx} = 0.0226\left[\frac{\mu}{\rho U}\right]^{\frac{1}{4}}\delta^{-\frac{1}{4}} \tag{3.6}$$

or

$$\delta^{\frac{1}{4}}d\delta = 0.2324\left[\frac{\mu}{\rho U}\right]^{\frac{1}{4}}dx \tag{3.7}$$

Integrating (3.7) and noting that the boundary layer is turbulent over the entire plate so that at *x=0*, the boundary layer thickness $\delta = 0$, we obtain after simplification

$$\delta^{\frac{5}{4}} = 0.2905\left[\frac{\mu}{\rho U}\right]^{\frac{1}{4}}x \tag{3.8}$$

so that

$$\delta = 0.3719\left[\frac{\mu}{\rho U}\right]^{\frac{1}{5}}x^{\frac{4}{5}} \tag{3.9}$$

or

$$\frac{\delta}{x} = 0.3719\left[\frac{\mu}{\rho U}\right]^{\frac{1}{5}} = \frac{0.3719}{(\mathrm{Re}_x)^{\frac{1}{5}}} \tag{3.10}$$

Another way of representing (3.12) is

$$\mathrm{Re}_\delta = 0.3719\,(\mathrm{Re}_x)^{\frac{1}{5}} \tag{3.11}$$

Equation (3.9) or (3.10) is the approximate turbulent boundary layer thickness.

3.1 Other Approximate Turbulent Characteristics Displacement Thickness δ_1

From Karman-Pohlhausen [8],

$$\delta_1 = \int_0^\delta \left(1 - \frac{u}{U}\right) dy \qquad (3.12)$$

Substituting (3.1) into (3.12) we find, after integration and simplification

$$\delta_1 = \int_0^\delta \left[1 - \left(\frac{y}{8}\right)^{\frac{1}{7}}\right] dy = \frac{1}{8}\delta \qquad (3.13)$$

Applying (3.10) in (3.13) yields

$$\frac{\delta_1}{x} = \frac{0.0463}{(\mathrm{Re}_x)^{1/5}} \qquad (3.14)$$

Momentum Thickness δ_2

Again, from K. Pohlhanan [8] ,

$$\delta_2 = \int_0^\delta \frac{u}{U}\left(1 - \frac{u}{U}\right) dy \qquad (3.15)$$

Substituting (3.1) in (3.15) we have

$$\delta_2 = \int_0^\delta \left(\frac{y}{\delta}\right)^{\frac{1}{7}} \left[1 - \left(\frac{y}{\delta}\right)^{\frac{1}{7}}\right] dy$$

Which on integration and simplification gives

$$\delta_2 = \frac{7}{72}\delta \qquad (3.16)$$

Using (3.10) in (3.16) leads to

$$\frac{\delta_2}{x} = \frac{0.03606}{(\mathrm{Re}_x)^{1/5}} \qquad (3.17)$$

Shape Factor H^*

Applying (3.14) and (3.17) we find

$$H^* = \frac{\delta_1}{\delta_2} = 1.2839 \qquad (3.18)$$

Wall Shear Stress τ_{approx}.

Substituting the value of δ from (3.10) into (2.4) we find

$$\tau_{approx} = 0.0226\, \rho U^2 \left[\frac{\mu}{\rho U \times \frac{0.3719\, x}{(\mathrm{Re}_x)^{1/5}}}\right] \qquad (3.19)$$

Which after simplification gives

$$\tau_{approx.} = 0.0288\, \rho U^2 \left[\frac{(\mathrm{Re}_x)^{1/5}}{\mathrm{Re}_x}\right]^{\frac{1}{4}}$$

or

$$\tau_{approx.} = 0.0288 \rho U^2 (\mathrm{Re}_x)^{-\frac{1}{5}} \qquad (3.20)$$

Skin Friction Coefficient C_f

By definition (Schlichting, [10])

$$C_f = \frac{\tau_{approx}}{\frac{1}{2}\rho U^2} \tag{3.21}$$

Substituting (3.20) in(3.21), we find after simplification

$$C_f = \frac{0.0576}{(\mathrm{Re}_x)^{\frac{1}{5}}} \tag{3.22}$$

Comparison of the approximate value (3.10) with the exact value (2.6) shows that the error in the boundary layer thickness is about 2.6%. Similarly, the error in the shape factor, by comparing the approximate value (3.18) with the exact value (2.7), is about 1.2 – 8.2%. These are adequate for the present purpose.

4.0 Illustrative Example

By considering selected values of Reynolds number typical of turbulent boundary layer flow, viz: $\mathrm{Re} = 10^6$, $2x10^6$, $3x10^6$, $4x10^6$ and $5x10^6$, and using these in (2.6), (3.10), (3.14), (3.17) and (3.22) the approximate values for these turbulent characteristics are determined and compared with the Blasius value of turbulent boundary layer thickness. The result is displayed in Table 1.

Table 1: *Reynolds Numbers and Blasius values with Approximate vaues of Characteristics*

Reynolds Number (Re)	Blasius value	Approximate Values of Characteristics			
	$\frac{\delta_{Blasius}}{x}$	$\frac{\delta}{x}$	$\frac{\delta_1}{x}$	$\frac{\delta_2}{x}$	C_f
10^6	$2.4x10^{-2}$	$2.3x10^{-2}$	$2.9x10^{-3}$	$2.2x10^{-3}$	$3.6x10^{-3}$
$2x10^6$	$2.0x10^{-2}$	$2.0x10^{-2}$	$2.5x10^{-3}$	$1.9x10^{-3}$	$3.1x10^{-3}$
$3x10^6$	$1.9x10^{-2}$	$1.8x10^{-2}$	$2.3x10^{-3}$	$1.8x10^{-3}$	$2.9x10^{-3}$
$4x10^6$	$1.8x10^{-2}$	$1.7x10^{-2}$	$2.2x10^{-3}$	$1.7x10^{-3}$	$2.7x10^{-3}$
$5x10^6$	$1.7x10^{-2}$	$1.7x10^{-2}$	$2.1x10^{-3}$	$1.6x10^{-3}$	$2.6x10^{-3}$

From Table 1, the graphs of the approximate characteristics and that of the exact Blasius boundary layer thickness versus Reynolds number are displayed in Figure 1.

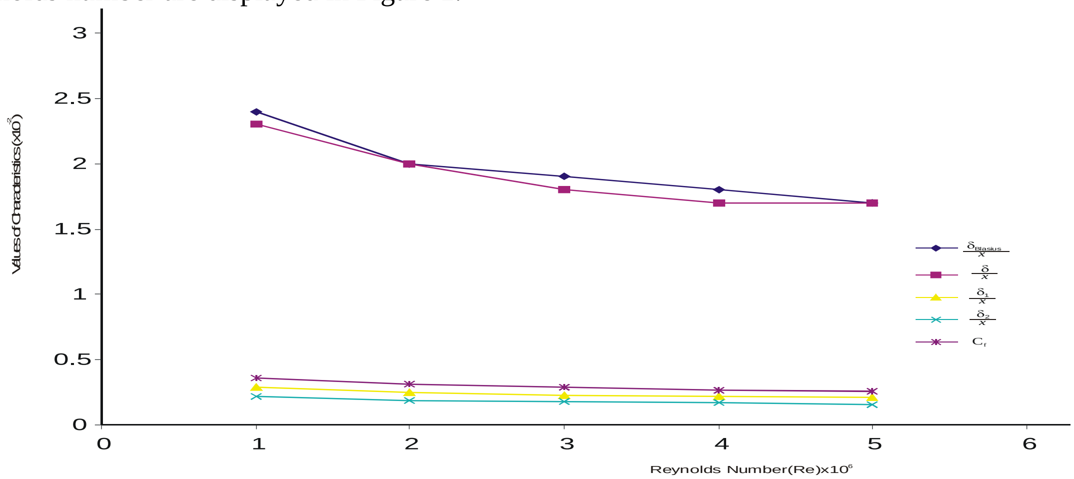

Figure 1: Graph of Blasius and approximate value of characteristics vs. Reynolds number (Re)

Discussion and Conclusion

This work uses a one-seventh power law in the momentum integral equation based on a turbulent boundary layer development along a smooth flat plate at zero incidence and zero pressure gradient, together with Blasius shear stress relation to determine the approximate value of the turbulent boundary layer thickness. Comparison of the approximate value (3.10) with the exact value (2.6) shows that the error in the turbulent boundary layer thickness is 2.6%. Similarly, the error in the shape factor, by comparing the approximate value (3.18) with the exact value (2.7), is about 1.2 – 8.2%. These results are adequate for the present purpose. In (3.10), we observe that as the boundary layer flow becomes more turbulent (i.e Re >> 5×10^5), the boundary layer thickness decrease with the consequent decrease in the percentage error. On the other hand, if the boundary layer thickness is expressed as in (3.11), we notice that Re_δ is large when Re_x is large and vice versa.

It is observe from Figure 1 that within the turbulent layer, the curves of the Blasius boundary layer thickness and that of the approximate boundary layer thickness get close to each other, coinciding with each other at the point $\mathrm{Re} = 5 \times 10^6$, whereas the curves of the displacement thickness, momentum thickness and skin friction coefficient get far away from that of Blasius boundary layer thickness.

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