Statistical Inferences on Type I and Type II Triangular Distributions

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Abstract

If continuous random variables X has a triangular distribution and if its boundary values are unknown, then it may well be necessary to make some statistical inferences related to these parameters. For triangular distributions, with unknown boundary values, some unbiased estimators are proposed. The proposed estimators are compared based on their efficiencies and the best estimators are determined and used for further inferences, such as constructing a confidence interval for , and the tests of hypotheses. By means of a simulation process, matching accuracy between sampling results and theoretical findings is observed.

Key Words: Triangular distribution, Estimators, Ordered statistics, Efficiencies of estimators, simulation

INTRODUCTION

1. Type I Triangular Distribution

If a continuous random variable X has the following probability density function (pdf), its distribution will be named as Type I Triangular distribution.

$$f(x) = \frac{2}{(\theta - a)^2}(x-a), a < x < \theta, \text{ (a is a known constant)} \qquad (1)$$

As it is shown below, all the moments of this distribution are the functions of the same parameter θ (upper bound value of X). If θ is unknown, then it must be estimated.

$$E(X^k) = \int_a^{\theta} x^k \frac{2}{(\theta-a)^2}(x-a)dx = \frac{2}{(\theta-a)^2}\left[\frac{x^{k+2}}{k+2} - \frac{ax^{k+1}}{k+1}\right]\Big|_{x=a}^{\theta}$$

$$E(X^k) = \frac{2}{(\theta-a)^2}\left[\frac{\theta^{k+2} - a^{k+2}}{k+2} - \frac{a\theta^{k+1} - a^{k+2}}{k+1}\right] \qquad (2)$$

If we let $Y = (X - a)$, then the distribution of Y turns out to be a special Type I triangular distribution. Y assumes values in the interval $(0, \lambda = (\theta - a))$ and its pdf will be as given in (4)

$$F(y) = P(Y \le y) = P(X \le a + y) = \int_a^{a+y} \frac{2}{(\theta-a)^2}(x-a)dx = \frac{2}{(\theta-a)^2}\left[\frac{x^2}{2} - ax\right]\Big|_{x=a}^{a+y}$$

$$F(y) = \frac{y^2}{(\theta-a)^2}, 0 < y < (\theta - a) = \lambda \qquad (3)$$

$$f(y) = \frac{2y}{\lambda^2}, 0 < y < \lambda \qquad (4)$$

1.1. Estimation of the parameter λ of the random Variable $Y=(X-a)$ by the Largest ordered Statistics $Y_{(n)}$

Let Ordered statistics obtained from a random sample of size n, taken from the pdf given in (4), be $Y_{(1)}<Y_{(2)}<...<Y_{(n)}$.

The pdf of $Y_{(n)}$ is as given below.

$$f_{Y_{(n)}}(y)=n[F(y)]^{n-1}f(y)=n\left[\frac{y^2}{\lambda^2}\right]^{n-1}\frac{2y}{\lambda^2}=\frac{2n}{\lambda^{2n}}y^{2n-1},0<y<\lambda \qquad (1.1.1)$$

The expected value and variance of $Y_{(n)}$ are as follows.

$$E(Y_{(n)})=\frac{2n}{\lambda^{2n}}\int_0^{\lambda} yy^{2n-1}dy=\frac{2n}{\lambda^{2n}}\left[\frac{y^{2n+1}}{2n+1}\Big|_0^{\lambda}\right]=\frac{2n\lambda}{(2n+1)} \qquad (1.1.\ 2)$$

$$E(Y_{(n)}^2)=\frac{2n}{\lambda^{2n}}\int_0^{\lambda} y^2y^{2n-1}dy=\frac{2n}{\lambda^{2n}}\left[\frac{y^{2n+2}}{2n+2}\Big|_0^{\lambda}\right]=\frac{2n\lambda^2}{(2n+2)} \qquad (1.1.\ 3)$$

$$Var(Y_{(n)})=\frac{2n\lambda^2}{(2n+2)}-\frac{4n^2\lambda^2}{(2n+1)^2}=\frac{n\lambda^2}{(n+1)(2n+1)^2} \qquad (1.1.\ 4)$$

Since, $Y=(X-a)$ and $\lambda=(\theta-a)$ then

$$E(X_{(n)})=E(Y_{(n})+a=\frac{2n\lambda}{(2n+1)}+a=\frac{2n\theta+a}{(2n+1)} \qquad (1.1.\ 5)$$

$$Var(X_{(n)})=Var(Y_{(n)})==\frac{n\lambda^2}{(n+1)(2n+1)^2}=\frac{n(\theta-a)^2}{(n+1)(2n+1)^2} \qquad (1.1.\ 6)$$

By the use of equation (1.1.5) we can obtain an unbiased estimator for the parameter θ of the random variable X. Hence,

$$T_1=\frac{(2n+1)X_{(n)}-a}{2n} \qquad (1.1.\ 7)$$

$$Var(T_1)=\frac{(2n+1)^2}{4n^2}[Var(X_{(n)})]=\frac{(\theta-a)^2}{4n(n+1)} \qquad (1.1.\ 8)$$

1.2. Estimation of the Parameter θ of the Random Variable X by the Sample Mean $\overline{X}$

We know that, the pdf of Y is $f(y)=\frac{2y}{\lambda^2},0<y<\lambda$, then

$$E(Y)=\int_0^{\lambda} y\frac{2y}{\lambda^2}dy=\frac{2y^3}{3\lambda^2}\Big|_0^{\lambda}=\frac{2\lambda}{3} \qquad (1.2.1)$$

$$E(Y^2)=\int_0^{\lambda} y^2\frac{2y}{\lambda^2}dy=\frac{2y^4}{4\lambda^2}\Big|_0^{\lambda}=\frac{\lambda^2}{2} \qquad (1.2.2)$$

$$Var(Y)=\frac{\lambda^2}{2}-\frac{4\lambda^2}{9}=\frac{\lambda^2}{18} \qquad (1.2.3)$$

Since $Y=(X-a)$, $E(X)=E(Y)+a$, and $Var(X)=Var(Y)$.

$$E(X)=\frac{2\lambda}{3}+a=\frac{2(\theta-a)}{3}+a=\frac{2\theta+a}{3} \qquad (1.2.4)$$

$$Var(X)=\frac{\lambda^2}{18}=\frac{(\theta-a)^2}{18} \qquad (1.2.5)$$

For any random variable X, $E(\overline{X})=E(X)$, and $Var(\overline{X})=\frac{Var(X)}{n}$. Then for above Type I Triangular distribution the following hold true.

$$E(\overline{X})=\frac{2\theta+a}{3}, \text{ and } Var(\overline{X})=\frac{(\theta-a)^2}{18n} . \qquad (1.2.6)$$

An unbiased estimator for θ as a function of $\overline{X}$ is as given below.

$T_2=\frac{3\overline{X}-a}{2}$ and its variance is $Var(T_2)=\frac{9}{4}Var(\overline{X})=\frac{(\theta-a)^2}{8n}$.

1.3. Comparisons of Unbiased Estimators on the Basis of Efficiency

$$T_1=\frac{(2n+1)X_{(n)}-a}{2n} \quad , \quad Var(T_1)=\frac{(\theta-a)^2}{4n(n+1)}$$

$$T_2=\frac{3\overline{X}-a}{2} \quad , \quad Var(T_2)=\frac{(\theta-a)^2}{8n}$$

For any integer n>1, $Var(T_1)<Var(T_2)$. Hence $T_1=\frac{(2n+1)X_{(n)}-a}{2n}$ is preferred over $T_2=\frac{3\overline{X}-a}{2}$.

1.4. Confidence Interval for θ of the random Variable X

The better unbiased estimator for θ is $T_1=\frac{(2n+1)X_{(n)}-a}{2n}$ and it is a linear function of $X_{(n)}$.

Hence, in the construction of a confidence interval it may be reasonable to use the pdf $X_{(n)}$.

From the pdf of X, $f(x)=\frac{2}{(\theta-a)^2}(x-a), a<x<\theta$, the following are obtained.

$$F(x)=\int_a^x \frac{2}{(\theta-a)^2}(y-a)dy=\frac{2}{(\theta-a)^2}\left[\frac{x^2-a^2}{2}-ax+a^2\right]=\frac{(x-a)^2}{(\theta-a)^2} \qquad (1.4.1)$$

The pdf of $X_{(n)}$ is as given below.

$$f_{X_{(n)}}(x)=n\left[\frac{(x-a)^2}{(\theta-a)^2}\right]^{n-1}\frac{2}{(\theta-a)^2}(x-a)=\frac{2n}{(\theta-a)^{2n}}(x-a)^{2n-1}, a<x<\theta \qquad (1.4.2)$$

From the probability statement, $P(x_{nL}<X_{(n)}<x_{nU})=1-\alpha$, lower and upper confidence limits may be obtained.

$$P(X_{(n)} \le x_{nL}) = \frac{2n}{(\theta-a)^{2n}} \int_a^{x_{nL}} (x-a)^{2n-1} dx = \alpha/2 \qquad (1.4.3)$$

In (1.4.3) let $x-a=t$: then $dt = dx; \quad x = a \to t = 0$; $x = x_{nL} \to t = x_{nL} - a$.

$$P(X_{(n)} \le x_{nL}) = \frac{2n}{(\theta-a)^{2n}} \int_a^{x_{nL}} (x-a)^{2n-1} dx = \frac{2n}{(\theta-a)^{2n}} \int_0^{x_{nL}-a} (t)^{2n-1} dt =$$

$$\frac{2n}{(\theta-a)^{2n}} \frac{t^{2n}}{2n} \Big|_{t=0}^{x_{nL}-a} = \left(\frac{x_{nL}-a}{\theta-a}\right)^{2n} = \frac{\alpha}{2}$$

x_{nL} is computed as given in (1.4.4).

$$x_{nL} = a + (\theta-a)\left(\frac{\alpha}{2}\right)^{1/2n} \qquad (1.4.4)$$

Similarly, the upper confidence limit is obtained as in (4.5).

$$P(X \le x_{nU}) = \frac{2n}{(\theta-a)^{2n}} \int_a^{x_{nU}} (x-a)^{2n-1} dx = \frac{2n}{(\theta-a)^{2n}} \int_0^{x_{nU}-a} (t)^{2n-1} dt =$$

$$\frac{2n}{(\theta-a)^{2n}} \frac{t^{2n}}{2n} \Big|_{t=0}^{x_{nU}-a} = \left(\frac{x_{nU}-a}{\theta-a}\right)^{2n} = 1-\frac{\alpha}{2}$$

$$x_{nU} = a + (\theta-a)\left(1-\frac{\alpha}{2}\right)^{1/2n} \qquad (1.4.5)$$

If (1.4.4) and (1.4.5) are substituted in the probability statement, $P(x_{nL} < X_{(n)} < x_{nU}) = 1-\alpha$, the following is obtained.

$$P\left(a + (\theta-a)\left(\frac{\alpha}{2}\right)^{1/2n} < X_{(n)} < a + (\theta-a)\left(1-\frac{\alpha}{2}\right)^{1/2n}\right) = 1-\alpha \qquad (1.4.6)$$

If the inequalities in (1.4.6) are solved for θ simultaneously, we reach the following expression.

$$P\left(a + \frac{X_{(n)}-a}{(1-\alpha/2)^{1/2n}} < \theta < a + \frac{X_{(n)}-a}{(\alpha/2)^{1/2n}}\right) = 1-\alpha \qquad (1.4.7)$$

Hence, a $100*(1-\alpha)\%$ confidence interval for θ can be given by the use of the following formula.

$$\left(a + \frac{X_{(n)}-a}{(1-\alpha/2)^{1/2n}}, \qquad a + \frac{X_{(n)}-a}{(\alpha/2)^{1/2n}}\right) \qquad (1.4.8)$$

1.5. Tests of Hypotheses Related to the Parameter θ of Type I Triangular

In testing $H_0 : \theta = \theta_0$ against to any alternative hypothesis, $X_{(n)} = X_{Max}$ may be used as a proper test statistic. For the chosen level of significance α, the decision rules given in the following table are applicable.

$H_0: \theta = \theta_0$ $H_1: \theta \neq \theta_0$	$H_0: \theta \leq \theta_0$ $H_1: \theta > \theta_0$	$H_0: \theta \geq \theta_0$ $H_1: \theta < \theta_0$
If $x_{(n)} \geq x_{nU}$ or $x_{(n)} \leq x_{nL}$ H_0 is rejected	If $x_{(n)} \geq x_{nU}$ H_0 is rejected	If $x_{(n)} \leq x_{nL}$ H_0 is rejected
Don't reject H_0 otherwise	Don't reject H_0 otherwise	Don't reject H_0 otherwise
Where, $x_{nL} = a + (\theta_0 - a)\left(\frac{\alpha}{2}\right)^{1/2n}$ and $x_{nU} = a + (\theta_0 - a)\left(1 - \frac{\alpha}{2}\right)^{1/2n}$	Where, $x_{nU} < a + (\theta_0 - a)(1-\alpha)^{1/2n}$	Where, $x_{nL} = a + (\theta_0 - a)(\alpha)^{1/2n}$

2. Statistical Inferences Related to Type II Triangular Distribution

If a random variable X has the following pdf, it is said to have a Type II triangular distribution.

$$f(x) = \frac{2}{(a-\theta)^2}(a-x), \theta < x < a \qquad (2.1)$$

All the moments of this distribution are functions of the parameter θ. For this reason, θ need to be estimated.

$$E(X^k) = \int_\theta^a x^k \frac{2}{(a-\theta)^2}(a-x)dx = \frac{2}{(a-\theta)^2}\left[\frac{ax^{k+1}}{k+1} - \frac{x^{k+2}}{k+2}\right]\Big|_{x=\theta}^{a}$$

$$E(X^k) = \frac{2}{(a-\theta)^2}\left[\frac{a^{k+2} - a\theta^{k+1}}{k+1} - \frac{a^{k+2} - \theta^{k+2}}{k+2}\right] \qquad (2.2)$$

If we let $Y = (X - \theta)$, then the distribution of Y turns out to have a special Type II triangular distribution. Y assumes values in the interval $(0, \lambda = (a-\theta))$ and its pdf will be as given in (2.3).
Estimation of $\lambda = (a-\theta)$ will enable to estimate θ.
.

$$F(y) = P(Y \leq y) = P(X \leq \theta + y) = \int_\theta^{\theta+y} \frac{2}{(a-\theta)^2}(a-x)dx = \frac{2}{(a-\theta)^2}\left[ax - \frac{x^2}{2}\right]\Big|_{x=\theta}^{\theta+y}$$

$$F(y) = \frac{2(a-\theta)y - y^2}{(a-\theta)^2} = \frac{2\lambda y - y^2}{\lambda^2}, 0 < y < (a-\theta) = \lambda \qquad (2.3)$$

$$f(y) = \frac{2(a-\theta) - 2y}{\lambda^2} = \frac{2(\lambda - y)}{\lambda^2}, 0 < y < \lambda \qquad (2.4)$$

2.1. Estimation of the parameter λ of the random Variable by the First ordered Statistics $Y_{(1)}$

Let Ordered statistics obtained from a random sample of size n, taken from the pdf given in (2.4), be $Y_{(1)} < Y_{(2)} < ... < Y_{(n)}$.
The pdf of $Y_{(1)}$ is as given below.

$$f_{Y_{(1)}}(y) = n[1-F(y)]^{n-1} f(y) = n\left[1-\frac{2\lambda y - y^2}{\lambda^2}\right]^{n-1} \frac{2(\lambda - y)}{\lambda^2} = \tag{2.1.1}$$

$$\frac{2n}{\lambda^{2n}}(\lambda - y)^{2n-1}, 0 < y < \lambda$$

The expected value and variance of $Y_{(1)}$ are as follows.

$$E(Y_{(1)}) = \int_0^\lambda y \frac{2n(\lambda - y)^{2n-1}}{\lambda^{2n}} dy \tag{2.1.2}$$

In (2.1.2), if we let $(\lambda - y) = t$:

$dt = -dy; \qquad y = 0 \rightarrow t = \lambda; \; y = \lambda \rightarrow t = 0$.

$$E(Y_{(1)}) = \int_0^\lambda y \frac{2n(\lambda - y)^{2n-1}}{\lambda^{2n}} dy = \frac{2n}{\lambda^{2n}} \int_0^\lambda (\lambda - t) t^{2n-1} dt =$$

$$\frac{2n}{\lambda^{2n}}\left[\frac{\lambda t^{2n}}{2n} - \frac{t^{2n+1}}{2n+1} \Big|_{t=0}^{\lambda}\right] = 2n\lambda\left[\frac{1}{2n} - \frac{1}{2n+1}\right] = \frac{\lambda}{2n+1}$$

$$E(Y_{(1)}) = \frac{\lambda}{2n+1} \tag{2.1.3}$$

$$E(Y_{(1)}^2) = \int_0^\lambda y^2 \frac{2n(\lambda - y)^{2n-1}}{\lambda^{2n}} dy = \frac{2n}{\lambda^{2n}} \int_0^\lambda (\lambda - t)^2 t^{2n-1} dt =$$

$$\frac{2n}{\lambda^{2n}}\left[\frac{\lambda t^{2n}}{2n} - \frac{2\lambda t^{2n+1}}{2n+1} + \frac{t^{2n+2}}{2n+2} \Big|_{t=0}^{\lambda}\right] = 2n\lambda^2\left[\frac{1}{2n} - \frac{2}{2n+1} + \frac{1}{2n+2}\right] = \frac{\lambda^2}{(2n+1)(n+1)}$$

$$E(Y_{(1)}^2) = \frac{\lambda^2}{(2n+1)(n+1)} \tag{2.1.4}$$

By the use of the results in (6.1.3) and (6.1.5) we obtain

$$Var(Y_{(1)}) = \frac{\lambda^2}{(n+1)(2n+1)} - \frac{\lambda^2}{(2n+1)^2} = \frac{n\lambda^2}{(n+1)(2n+1)^2} \tag{2.1.5}$$

Since $Y = (X - \theta)$,

$$E(X_{(1)}) = E(Y_{(1)}) + a = \frac{\lambda}{(2n+1)} + \theta = \frac{2n\theta + a}{(2n+1)} \tag{2.1. 6}$$

$$Var(X_{(1)}) = Var(Y_{(1)}) == \frac{n\lambda^2}{(n+1)(2n+1)^2} = \frac{n(a-\theta)^2}{(n+1)(2n+1)^2} \tag{2.1.7}$$

From the equation (2.1.6), we can obtain an unbiased estimator for θ as a function of $X_{(1)}$ as given in (2.1.8) and the variance of T_1 are obtained as in (2.1.9).

$$T_1 = \frac{(2n+1)X_{(1)} - a}{2n} \tag{2.1.8}$$

$$Var(T_1) = \frac{(2n+1)^2}{4n^2} Var(X_{(1)}) == \frac{(a-\theta)^2}{4n(n+1)} \tag{2.1.9}$$

2.2. Estimation of the Parameter θ of X by the Sample Mean $\overline{X}$

$$f(y) = \frac{2(a-\theta) - 2y}{\lambda^2} = \frac{2(\lambda - y)}{\lambda^2}, 0 < y < \lambda$$

$$E(Y) = \int_0^{\lambda} y \frac{2(\lambda - y)}{\lambda^2} dy = \frac{2}{\lambda^2}\left[\frac{\lambda y^2}{2} - \frac{y^3}{3} \Big|_{y=0}^{\lambda}\right] = \frac{2}{\lambda^2}\left[\frac{\lambda^3}{2} - \frac{\lambda^3}{3}\right] = \frac{\lambda}{3} \tag{2.2.1}$$

$$E(Y^2) = \int_0^{\lambda} y^2 \frac{2(\lambda - y)}{\lambda^2} dy = \frac{2}{\lambda^2}\left[\frac{\lambda y^3}{3} - \frac{y^4}{4} \Big|_{y=0}^{\lambda}\right] = \frac{2}{\lambda^2}\left[\frac{\lambda^4}{3} - \frac{\lambda^4}{4}\right] = \frac{\lambda^2}{6} \tag{2.2.2}$$

$$Var(Y) = \frac{\lambda^2}{6} - \frac{\lambda^2}{9} = \frac{\lambda^2}{18} \tag{2.2.3}$$

Since $Y = (X - \theta)$, $E(X) = E(Y) + \theta$, and $Var(X) = Var(Y)$.

$$E(X) = \frac{\lambda}{3} + \theta = \frac{(a-\theta)}{3} + \theta = \frac{2\theta + a}{3} \tag{2.2.4}$$

$$Var(X) = \frac{\lambda^2}{18} = \frac{(a-\theta)^2}{18} \tag{2.2.5}$$

For any random variable X, $E(\overline{X}) = E(X)$, and $Var(\overline{X}) = \frac{Var(X)}{n}$. Then for above Type II Triangular distribution the following hold true.

$$E(\overline{X}) = E(X) = \frac{2\theta + a}{3}, \text{ and } \quad Var(\overline{X}) = \frac{1}{n} Var(X) = \frac{(a-\theta)^2}{18n}.$$

An unbiased estimator for θ, as a function of $\overline{X}$, and its variance are as given below.

$$T_2 = \frac{3\overline{X} - a}{2} \quad . \tag{2.2.6}$$

$$Var(T_2) = \frac{9}{4} Var(\overline{X}) = \frac{(a-\theta)^2}{8n} \tag{2.2.7}$$

2.4. Comparisons of Unbiased Estimators on the Basis of Efficiency

$$T_1 = \frac{(2n+1)X_{(1)} - a}{2n} \quad , \quad Var(T_1) = \frac{(\theta - a)^2}{4n(n+1)}$$

$$T_2 = \frac{3\overline{X} - a}{2} \quad , \quad Var(T_2) = \frac{(\theta - a)^2}{8n}$$

For any n>1, $Var(T_1) < Var(T_2)$. $T_1 = \frac{(2n+1)X_{(1)} - a}{2n}$ is a better unbiased estimator.

2.5. Confidence Interval for θ of the Type II Triangular Distribution

Since we let $Y = (X - \theta)$ and its pdf is obtained in (2.1.1) as given below

$$f_{Y_{(1)}}(y) = \frac{2n}{\lambda^{2n}}(\lambda - y)^{2n-1}, 0 < y < \lambda \ .$$

From the probability statement, $P(x_{1L} < X_{(1)} < x_{1U}) = 1-\alpha$, lower and upper confidence limits may be obtained.

$$P(x_{1L} < X_{(1)} < x_{1U}) = P(x_{1L} < Y_{(1)} + \theta < x_{1U}) = P(x_{1L} - \theta < Y_{(1)} < x_{1U} - \theta) = 1-\alpha$$

If we let $(x_{1L} - \theta) = y_{1L}$ *and* $(x_{1U} - \theta) = y_{1U}$

$$P(Y_{(1)} < y_{1L}) = \frac{2n}{\lambda^{2n}}\int_0^{y_{1L}}(\lambda - y)^{2n-1}dy = \alpha/2 \qquad (2.5.1)$$

$$P(Y_{(1)} < y_{1U}) = \frac{2n}{\lambda^{2n}}\int_0^{y_{1U}}(\lambda - y)^{2n-1}dy = 1-\alpha/2 \qquad (2.5.2)$$

In (2.5.1) and (2.5.2), let $(\lambda - y) = t$:

$dt = -dy; \qquad y = 0 \to t = \lambda; \ y = y_{1L} \to t = \lambda - y_{1L}$

$$P(Y_{(1)} < y_{1L}) = \frac{2n}{\lambda^{2n}}\int_0^{y_{1L}}(\lambda - y)^{2n-1}dy = \frac{2n}{\lambda^{2n}}\int_{\lambda - y_{1L}}^{\lambda} t^{2n-1}dt = \left(\frac{\lambda}{\lambda - y_{1L}}\right)^{2n} = \alpha/2 \qquad (2.5.3)$$

Similarly,

$$P(Y_{(1)} < y_{1U}) = \frac{2n}{\lambda^{2n}}\int_0^{y_{1U}}(\lambda - y)^{2n-1}dy = \frac{2n}{\lambda^{2n}}\int_{\lambda - y_{1U}}^{\lambda} t^{2n-1}dt = \left(\frac{\lambda}{\lambda - y_{1U}}\right)^{2n} = 1-\alpha/2 \qquad (2.5.4)$$

From the equations (2.5.3) and (2.5.4), y_{1L} and y_{1U} can be computed as given below.

$$y_{1L} = \lambda - (\lambda)\left(\frac{\alpha}{2}\right)^{-1/2n} = (a-\theta) - (a-\theta)\left(\frac{\alpha}{2}\right)^{-1/2n} \qquad (2.5.5)$$

$$y_{1U} = \lambda - (\lambda)\left(1-\frac{\alpha}{2}\right)^{-1/2n} = (a-\theta) - (a-\theta)\left(1-\frac{\alpha}{2}\right)^{-1/2n} \qquad (2.5.6)$$

Since we have $(x_{1L} - \theta) = y_{1L}$ *and* $(x_{1U} - \theta) = y_{1U}$, then

$x_{1L} = y_{1L} + \theta$ and $x_{1U} = y_{1U} + \theta$.

$$x_{1L} = (a) - (a-\theta)\left(\frac{\alpha}{2}\right)^{1/2n} \qquad (2.5.7)$$

$$x_{1U} = (a) - (a-\theta)\left(1-\frac{\alpha}{2}\right)^{-1/2n} \qquad (2.5.8)$$

Substituting the results of (2.5.7) and (2.5.8) in the following probability expression yields (2.5.9).

$$P(x_{1L} < X_{(1)} < x_{1U}) = 1-\alpha$$

$$P\left(a-(a-\theta)\left(\frac{\alpha}{2}\right)^{-1/2n} < X_{(1)} < a-(a-\theta)\left(1-\frac{\alpha}{2}\right)^{-1/2n}\right)=1-\alpha \qquad (2.5.9)$$

By solving inequalities in (2.4.9) for θ simultaneously we obtain the following probability statement.

$$P\left(a-(a-X_{(1)})(1-\alpha/2)^{1/2n} < \theta < a-(a-X_{(1)})(\alpha/2)^{1/2n}\right)=1-\alpha \qquad (2.510)$$

Hence, a $100*(1-\alpha)\%$ confidence interval for θ can be given by the use of the following formula.

$$\left(a-(a-X_{(1)})(1-\alpha/2)^{1/2n}, \quad < a-(a-X_{(1)})(\alpha/2)^{1/2n}\right)$$

2.6. Tests of Hypotheses Related to the Parameter θ of Type II Triangular Distribution

In testing $H_0:\theta=\theta_0$ against to any alternative hypothesis, $X_{(1)} = X_{Min}$ may be used as a proper test statistic. For the chosen level of significance α, the decision rules given in the following table are applicable.

$H_0:\theta=\theta_0$ $H_1:\theta\neq\theta_0$	$H_0:\theta\leq\theta_0$ $H_1:\theta>\theta_0$	$H_0:\theta\geq\theta_0$ $H_1:\theta<\theta_0$
If $x_{(1)}\geq x_{1U}$ or $x_{(1)}\leq x_{1L}$ H_0 is rejected.	If $x_{(1)}\geq x_{1U}$ H_0 is rejected.	If $x_{(1)}\leq x_{1L}$ H_0 is rejected.
Don't reject H_0 otherwise.	Don't reject H_0 otherwise.	Don't reject H_0 otherwise.
Where, $x_{1L}=a-(a-\theta_0)\left(\frac{\alpha}{2}\right)^{1/2n}$ and $x_{1U}<a-(a-\theta_0)\left(1-\frac{\alpha}{2}\right)^{1/2n}$	Where, $x_{1U}<a-(a-\theta_0)(1-\alpha)^{1/2n}$	Where, $x_{1L}=a-(a-\theta_0)(\alpha)^{1/2n}$

3. Simulation

For the purpose of simulations from both Type I, and Type II triangular distributions inverse transformations are used on the respective F(x) functions.

For the type I triangular distribution over the interval (5,10) [$\theta=5$ *and* $a=10$] the pdf is

$f(x)=\dfrac{2(x-5)}{25}, 5<x<10$. Hence we obtain

$F(x)=\dfrac{(x-5)^2}{25}, 5<x<10$.

If a random sample is taken from the $Uniform[0,1]$ we obtain the corresponding X value in the type I triangular distribution by the use of the following formula

$x=5(1+\sqrt{F(x)})$.

The median, m, for this Type I triangular distribution is:

$$F(m)=\frac{(m-5)^2}{25}=\frac{1}{2}\rightarrow\frac{(m-5)}{5}=\frac{1}{\sqrt{2}}\rightarrow m=5+\frac{5}{\sqrt{2}}\cong 8.535$$

Similarly, for the type II triangular distribution over the interval (5, 10) **[** $a=5$ *and* $\theta=10$]

the pdf is

$f(x)=\frac{2(10-x)}{25}, 5<x<10$, and the respective distribution function is

$F(x)=1-\frac{(10-x)^2}{25}, 5<x<10$.

If a random sample is taken from the $Uniform[0,1]$ we obtain the corresponding X value in the Type II triangular distribution by the use of the following formula

$x=10-5\sqrt{1-F(x)}$.

The median, m, for this Type II triangular distribution is:

$F(m)=1-\frac{(10-m)^2}{25}=\frac{1}{2}\rightarrow\frac{(10-m)}{5}=\frac{1}{\sqrt{2}}\rightarrow m=10-\frac{5}{\sqrt{2}}\cong 6.464$.

The following tables are shortened forms of sample observations from Uniform(0,1) distribution and the respective sample observations from Type I and II triangular(5,10) distributions.

50 Random samples of sizes n=30, from Uniform (0, 1)

u_1	u_2	u_3	u_4	u_5	...	u_{48}	u_{49}	u_{50}
0,517788	0,498149	0,73755	0,231313	0,906975	...	0,357554	0,955998	0,259729
0,746209	0,260626	0,239369	0,473726	0,815237		0,171827	0,454524	0,764033
0,377066	0,449645	0,178527	0,453634	0,773392		0,483062	0,988522	0,711061
0,692188	0,387486	0,778406	0,122221	0,626263		0,261265	0,929536	0,805118
0,606739	0,406041	0,60322	0,43981	0,234233		0,759478	0,567909	0,919013
0,421063	0,729669	0,367489	0,722403	0,2267		0,750751	0,153411	0,369019
0,80357	0,124422	0,543059	0,808808	0,504451		0,324161	0,27868	0,516978

50 Random samples of sizes n=30, from TYPE I Triangular Distribution and Estimators: $x_i=5(1+\sqrt{u_i})$

	X1	***X2***	***X3***	***X4***	***X5***	....	***X49***	***X50***		
	8,597876	8,528984	9,294036	7,404752	9,761761	....	9,888757	7,548182		
	9,31917	7,552579	7,446267	8,441389	9,514525	.....	8,370919	9,37045		
	8,070286	8,352779	7,112625	8,367617	9,397135	.....	9,971222	9,216223		
	8,24447	9,271034	8,031045	9,249713	7,380651	.....	6,958389	8,037347		
	9,482104	6,763672	8,684627	9,496689	8,551237	.....	7,639509	8,595061	***Mean***	***Var***
Mean	**8,36463**	**8,312875**	**8,433408**	**8,721029**	**8,75695**	**.....**	**8,426896**	**8,771766**	**8,4317**	**0,0460**
Var	**1,296027**	**1,174865**	**0,980048**	**1,171179**	**0,798496**	**.....**	**1,739837**	**0,767044**	**1,2601**	**0,0940**
Xmax	**9,94276**	**9,89737**	**9,945686**	**9,860279**	**9,930852**	**.....**	**9,971222**	**9,864761**	**9,921**	**0,0044**
Xmin	**5,166093**	**6,20382**	**6,297411**	**5,459905**	**6,89126**	**.....**	**5,809413**	**6,676654**	**5,8748**	**0,2427**
T1	**10,02514**	**9,978993**	**10,02811**	**9,941284**	**10,01303**	**... ...**	**10,05408**	**9,94584**	**10,003**	**0,0045**
T2	**10,04695**	**9,969312**	**10,15011**	**10,58154**	**10,63542**	**... ...**	**10,14034**	**10,65765**	**10,147**	**0,1037**
Med	**8,672594**	**8,436987**	**8,651681**	**9,223838**	**8,953614**	**.....**	**8,842737**	**8,854906**	**8,6634**	**0,0948**
Tmed	**10,19383**	**9,860634**	**10,16426**	**10,97341**	**10,59125**	**... ...**	**10,43445**	**10,45166**	**10,180**	**0,1897**

The simulation results, given in the above and below tables, are in accordance with the theoretical findings, for that the estimator T_1 is seen to estimate the true parameter θ of the Type I and Type II triangular distribution very accurately.

50 Random samples 0f sizes n=30, from TYPE II Triangular Distribution

and Estimators: $x_i = 10 - 5\sqrt{1-u_i}$

	X1	*X2*	*X3*	*X4*	….	*X48*	*X49*	*X50*		
	6,527927	6,457928	7,438506	5,616261	….	5,992363	8,951165	5,69805		
	7,481117	5,700658	5,639291	6,372764	…..	5,449799	6,307182	7,571179		
	6,053693	6,290705	5,468244	6,30417	…..	6,405081	9,464323	7,312349		
					…..					
	6,195606	7,400334	6,023473	7,365624	…..	7,50376	5,399488	6,028285		
	7,783981	5,321382	6,62013	7,813727	…..	5,88953	5,753473	6,525013	***Mean***	***Var***
Mean	***6,651137***	***6,596583***	***6,683158***	***7,051849***	…..	***6,528217***	***6,903038***	***7,029369***	***6,767***	***0,0464***
Var	***1,088856***	***1,187989***	***1,158918***	***1,211741***	…..	***1,605636***	***1,807143***	***1,11921***	***1,335***	***0,0732***
Xmax	***9,245597***	***8,992148***	***9,265025***	***8,82625***	…..	***9,072844***	***9,464323***	***8,844968***	***9,194***	***0,1313***
Xmin	***5,002759***	***5,147082***	***5,171261***	***5,021196***	… …	***5,017774***	***5,06595***	***5,289498***	***5,102***	***0,0105***
T1	***5,002805***	***5,149533***	***5,174115***	***5,02155***	… …	***5,018071***	***5,067049***	***5,294323***	***5,104***	***0,0109***
T2	***4,976705***	***4,894875***	***5,024737***	***5,577773***	…..	***4,792326***	***5,354557***	***5,544054***	***5,151***	***0,1045***
Median	***6,607878***	***6,368598***	***6,584901***	***7,32477***	… …	***6,198722***	***6,803171***	***6,815711***	***6,630***	***0,1153***
Tmed	***5,202815***	***4,864422***	***5,170321***	***6,216653***		***4,624181***	***5,479002***	***5,496736***	***5,234***	***0,2306***

REFERENCES

Balakrishnan, N. and Cohen, A. C. *Order Statistics and Inference.* New York: Academic Press, 1991.

Balakrishnan, N. and Rao, C. R. (Eds.). *Handbook of Statistics, Vol. 16: Order Statistics: Theory and Methods.* Amsterdam, Netherlands: Elsevier, 1998.

Balakrishnan, N. and Rao, C. R. (Eds.). *Order Statistics: Applications.* Amsterdam, Netherlands: Elsevier, 1998.

David, H. A. *Order Statistics, 2nd ed.* New York: Wiley, 1981.

Gibbons, J. D. and Chakraborti, S. (Eds.). *Nonparametric Statistic Inference, 3rd ed. exp. rev.* New York: Dekker, 1992.

Hogg, R. V. and Craig, A. T. *Introduction to Mathematical Statistics, 3rd ed.* New York: Macmillan, 1970.

Rose, C. and Smith, M. D. "Order Statistics." §9.4 in *Mathematical Statistics with Mathematica.* New York: Springer-Verlag, pp. 311-322, 2002.

Rose, C. and Smith, M. D. "Computational Order Statistics." *Mathematica J.* **9**, 790-802, 2005.