Prediction of Orthometric Heights of Points for Topographic Mapping

T. O. Idowu
T. Abubakar
Department of Surveying and Geoinformatics
Modibbo Adama University of Technology
Yola, Nigeria.

Abstract

Heights are determined based on a particular reference surface for the purpose of controlling vertical coordinates of points during mapping. These heights are usually presented in various countries as orthometric heights which has geoid as its reference surface. Orthometric height is the geometric distance from a point on the earth's surface measured along the curved plumbline to the geoid. In practice, orthometric heights are determined by conventional precise levelling combined with gravity observations along the levelling routes. They have conceptual importance in surveying as they provide the needed heights for controlling other height measurements in surveying. However, its determination seems to be time consuming, laborious and less economical of project cost. Where the density of the available orthometric heights in a study area is not enough, more heights can be measured. In some cases, where observations are not possible, probably due to inaccessibility of the stations needed for measurement or logistic problems, the required orthometric heights can be estimated (predicted). Therefore, this paper attempts the prediction of orthometric heights for unknown stations in a study area using least squares collocation technique. Data used for the study are the rectangular coordinates and orthometric heights of thirty known stations within the study area. Analyzing the results obtained, it was observed that the least squares collocation technique used for the prediction did not introduce significant distortion into the results obtained at 5% level of significance. Also, it was observed that the predicted heights satisfactory. That is, the difference between the observed and the predicted orthometric heights of the same station was found to be within the tolerant error limit.

Key-words: Orthometric heights, Geoid, Vertical coordinates, Topographic Mapping and Least Squares collocation

1. Introduction

Mapping can be defined as the process of map-making. That is, mapping is a process that starts with collection and manipulation of geospatial data and ends with the creation of map through cartographic activities. The type of geospatial data needed for mapping depends on the required type of map. For instance, in topographic mapping, rectangular and vertical (height) coordinates are the data needed to produce a map that defines the topography of the earth's surface. Rectangular coordinates determine the planimetric positions of objects on the earth's surface while the vertical coordinates give the measures of elevation of points on the earth's surface with respect to a specified datum (reference surface). Generally, heights, as vertical coordinates, are of two types. These are Global Position System (GPS) heights and orthometric heights. GPS heights are the heights of points on the earth's surface measured with respect to the ellipsoid as reference surface. Its theory, demonstration, uses and benefits in surveying and geodesy are fully discussed in Gerdan (1991), Tiberius and Borre (2000) and Idowu (2007). However, its heights are not based on reference surface needed for the purpose of topographic mapping. Orthometric heights (and not GPS heights) are usually presented in various countries as vertical coordinates for mapping (Moka, 2011). They are the heights measured along curved plumbline from the surface of the earth to an imaginary reference surface called geoid. Geoid is an equipotential surface, usually approximated by the Mean Sea Level (MSL) and often determined using earth's gravity field. Therefore, for topographic mapping purpose, it is required to carry out the measurements of such heights. Where the density of the available orthometric heights in a project area is not enough, more heights can be measured.

In some cases, where measurements are not possible, probably due inaccessibility of points or logistic problems, the required orthometric heights can be predicted. Therefore, it is the objective of this paper to attempt the prediction of orthometric heights for points that cannot be determined by measurement using least squares collocation technique. Least squares collocation technique is preferred because, according to Idowu (2005) and Idowu (2006), it appears to be a better alternative to other methods of prediction.

2. Methodology

The stages of approach adopted includes the use of classical least squares technique to predict orthometric heights at the observation stations, the use of optimization technique to determine the optimal values of covariance parameters needed for the evaluation of covariance function and the use of least squares collocation technique for the prediction of orthometric heights at the observation stations and outside observation stations.

2.1 Data acquisition

The data used for the study are shown in tables 1 and 2. They are the results of field observations obtained from the University of Lagos, Nigeria. Columns 1, 2, 3 and 4 of the table show the station numbers, X-coordinates, Y-coordinates and orthometric heights respectively. The coordinates are based on rectangular coordinate system. Stations on table 1 represent thirty observation (known) stations while the fourteen stations in table 2 are regarded as outside observation (unknown) stations.

Table 1: Data used for predicting Orthometric heights at observation stations

Station no.	X(m)	Y(m)	Orthometric Height (m)
DOS07S	544147.210	720224.550	07.516784105
SGIS006	543167.610	719837.750	04.315235000
ED015	542688.050	720211.670	05.830372000
SGYM017	543395.790	720585.490	05.839751000
SGJA015	543953.290	720340.470	06.928477341
SGDL014	543505.140	719962.970	02.937411803
PGS9	543929.400	720012.190	07.020878584
XST347AZ	543774.000	720026.280	05.351150088
MEGA11	542593.340	720460.300	05.049271000
ED009	543277.130	720564.240	05.749291000
ED006	543896.920	720573.170	07.548386694
DSG07	544436.730	720547.220	01.641999968
XST347	543235.430	719894.220	04.701000000
SGEG02	544120.510	720370.540	01.815455359
USL01	544023.400	720623.510	08.479188863
SGEN012	542978.690	720445.370	04.972066000
SGUG007	543019.110	719778.390	03.807963000
DSG003	544230.510	720905.680	02.129748152
ED014	542757.210	720090.190	05.991619009
CR5	543307.880	720388.280	03.521918000
CR8	543240.370	719908.570	04.191252000
CBLM3	543750.530	720013.910	04.633840277
SGRN008	543070.730	719915.810	03.450022000
YTT28-186	542621.490	720383.370	06.265957000
SGCP016	543547.310	720553.880	07.414177278
DOS04S	543262.550	720603.430	05.575347000
PGD81-1	543641.680	720562.230	06.604817993
CR3	543309.710	720310.060	03.945358000
DOS03S	543099.440	720531.940	06.626035000
SGEG003	544134.740	720560.380	07.525433165

Table 2: Data used as Orthometric heights outside observation stations

Sation no.	X(m)	Y(m)	Orthometric Height (m)
SGEG01	544269.090	720409.100	02.530312838
ED011	543362.520	719930.750	03.382621938
SD14S	543300.930	720148.050	03.227390000
SGEN010	542772.920	720429.030	05.563834000
MEGA05	544481.890	720427.270	01.824800000
MEGA10	543076.800	720512.420	06.222928000
SGEG009	542586.010	720266.730	05.822543000
SGLF005	544326.760	720585.690	03.202049358
CR6	543310.170	720191.590	03.396077000
GMS05	544007.130	720597.640	08.247086872
SGEN011	542924.060	720445.500	04.977269000
MEGA04	542671.790	720209.490	05.872013000
DOS14S	542584.260	720382.310	05.883506000
ED013	542885.190	720002.910	05.075428000

2.2 Data Quality

The quality of data used in any experiment can be determined by the validity and reliability of such data. The validity is measured by the precision while the reliability is determined by the accuracy of the data. Details of these are discussed in Idowu (2005). The quality control test for the data used in this study was carried out by the authors and Department of Surveying and Geoinformatics, University of Lagos and the results showed that the validity, reliability and hence quality of the data are satisfactory.

2.3 Data Processing

2.3.1 Least Squares Collocation Technique

The detailed concepts and derivation of equations of the least squares collocation technique are discussed in Krarup (1970), Moritz (1972), Krakiwsky (1975), Rapp (1986), Ezeigbo (1988) and Ayeni (2001). However, for easy reference, the step-by-step applications of the equations are summarized in Idowu (2006). The applications of the summary, for this study, show the least squares collocation model given in equation (1) for the prediction of signals (orthometric heights) and filtering of errors without the determination of any parameter.

$$H = h + e \tag{1}$$

The condition for the solution of equation (1) is given as equation (2)

$$h^T C_{hH}^{-1} h + e^T C_{ee}^{-1} e = \text{minimum} \tag{2}$$

Where H = Observed Orthometric Heights

$h = C_{hH} C_{HH}^{-1} H$ (Predicted Orthometric Heights (Signals))

$e = C_{ee} C_{HH}^{-1} H$ (Errors vectors of the Signals)

$C'_{hh} - C_{hH} C_{HH}^{-1} C^T_{hH}$ (Error Covariance matrix of signals)

C_{hH} = Covariance matrix which shows degree of correlation between observations and signals

C_{ee} = Error covariance matrix of the observations

C_{HH} = Covariance matrix which shows degree of correlation between observations

Covariance function plays a significant role in the concept of the least squares collocation technique. Therefore, it should be simple, analytical, isotropic and homogeneous (Schwarz, 1976, Moritz, 1978 and Idowu, 2006). There are various mathematical expressions for the covariance functions that represent global features of the earth. Detailed discussions on these can be found in Moritz (1972), Schwarz (1976a), Moritz (1978), Rapp (1986) and Ezeigbo (1988). However, for some limited purposes such as prediction of orthometric heights within a flat survey area, one may approximate the curved surface of the earth locally by a plane surface. Therefore, the best fit mathematical expression for covariance function is given by Idowu (2005) as equation (3).

$$C(r) = C(o)/(1 + (r/a)^2)^{1/2} \quad (3)$$

Where a = correlation length

$C(r)$ = covariance as a function of horizontal distance (r) between two stations

$C(o)$ = covariance function at zero distance (i. e. one station where r = 0)

Three parameters, C(o), ‘a’ and ‘r’, are needed to evaluate C(r). The value of ‘r’ is considered known since it can be measured or computed from the rectangular coordinates of the stations. However, the values of C(o) and ‘a’ are unknown hence they are to be estimated by optimization technique. Optimization procedure systematically searches among the range of possible values of parameters and selects the best-fit values which satisfy the given objective function. In this study, optimization technique is used to determine the optimal values of C(o) and ‘a’ for the evaluation of C(r) using the objective function given as:

$$Q^2 = (F_o^2 - F^2)/F_0^2 \quad (4)$$

Where: $F_o^2 = \Sigma_{i=1}^{n}(H_i - H_m)^2$

$F^2 = \Sigma_{i=1}^{n}(H_i - h_i)^2$

H_i = observed orthometric heights at point i .

h_i = predicted orthometric heights at point i.

H_m = mean orthometric height

Optimal values of the parameters are obtained when F^2 is a minimum and Q^2 is approximately equal to 1.

The optimization process starts by using classical least squares technique to solve equation (5) to obtain the values of v_i.

$$H_i = h_i + v_i \quad (5)$$

In order to achieve this, The values of h_i are represented by equation (6).

$$h_i = b_0 + b_1 H_i + b_2 H_i^2 \quad (6)$$

Where: b_0, b_1, b_2 = constant coefficients

Putting (6) in equation (5), the values of v_i are determined by classical least squares method. Thereafter, the estimates of C(o) and ‘a’ are determined by solving equations (7) and (8) as in Idowu (2006):

$$\sum_{i=1}^{n-k} v_i v_{i+k} = X \quad (7)$$

$$X = df\, C(o)/(1 + (r/a)^2)^{1/2} \quad (8)$$

Where: df = degree of freedom = n - m

m = number of parameters

n = number of observations

k = 0, 1, 2, 3, ..., n-1

i = 1, 2, 3, ..., n-k

The results, that is, the values of C(o) and ‘a’ are given as equations (9), (10) and (11).

$$C(o) = \sum_{i=1}^{n} v_i v_i/df \quad (9)$$

$$a_i = (r^2/((C(o)/\sum_{i=1}^{n-k} v_i v_{i+k})^2 - 1))^{1/2} \quad (10)$$

$$a = \sum_{i=1}^{n} a_i/n \quad (11)$$

Thereafter, the covariance parameters obtained are used to evaluate C(r) for the prediction of h_i by least squares collocation technique. The predicted values are expressed as shown in equation (12).

$$h_i = C_{hH}\, C_{HH}^{-1} H \quad (12)$$

The values of h_i and H_i are then used in equation (4) to compute the values of F_i^2. Thereafter, the values of the covariance parameters are allowed to vary systematically and used to compute the values of F_{i+1}^2. For a successful process, F_{i+1}^2 must be less than F_i^2. This is an iterative process which continues until F_{i+1}^2 is a minimum. Therefore, the optimal values of the parameters are obtained when F_{i+1}^2 is a minimum and Q^2 is approximately equal to 1. It is pertinent to note that the results obtained for each iteration include h_i, C_0 and ‘a’. Therefore, in the iterative process, the production of the optimal values of the parameters (C_0 and a) leads to the production of the equivalent values of the required predicted orthometric heights (h_i). A computer program written in Fortran 77 language was used for all the computations. The results obtained are presented in the next section.

3. Presentation of result

The extract of the searched covariance parameters used for the prediction of the orthometric heights are shown in table 3. Row 11 of this table shows the optimal values of the covariance parameters. Results of the predicted orthometric heights at the observation stations are shown in table 4 while table 5 shows the results of the predicted orthometric heights outside the observation stations. Columns 1 to 4 of tables 4 and 5 are the station numbers, observed orthometric heights, predicted orthometric heights and the difference (E) between the observed and the predicted orthometric heights respectively. The parameters used for the statistical analysis of the results are shown in table 6. These include degree of freedom, upper limit of the table statistics, computed statistics and the lower limit of table statistics.

Table 3: Searched covariance parameters for the prediction of Orthometric heights

C(o)	a	F^2	R^2
309.299866	175.599350	0.36112178595831000000	0.99112693327535100000
310.499866	175.809350	0.36086556342511800000	0.99113322887899900000
311.699866	176.019350	0.36063530549997200000	0.99113888651033900000
312.899866	176.229350	0.36043103567391100000	0.99114390559217500000
314.099866	176.439350	0.36025277742448800000	0.99114828554764400000
315.299866	176.649350	0.36010055421155100000	0.99115202580031800000
316.499866	176.859350	0.35997438947639400000	0.99115512577422400000
317.699866	177.069350	0.35987430663750600000	0.99115758489395000000
318.899866	177.279350	0.35980032909041400000	0.99115940258464900000
320.099866	177.489350	0.35975248020516800000	0.99116057827209800000
321.299866	***177.699350***	***0.35973078332465100000***	***0.99116111382744000000***
322.499866	177.909350	0.35973526175897200000	0.99116100134383700000
323.699866	178.119350	0.35976593878751500000	0.99116024758338400000
324.899866	178.329350	0.35982283765363300000	0.99115884953027500000
326.099866	178.539350	0.35990598156632500000	0.99115680661424500000
327.299866	178.749350	0.36001539369288100000	0.99115411826605600000
328.499866	178.959350	0.36015109716242600000	0.99115078391740400000
329.699867	179.169350	0.36031311505807400000	0.99114680300111900000
330.899867	179.379350	0.36050147041900700000	0.99114217495110800000
332.099867	179.589350	0.36071618623781300000	0.99113689920242400000
333.299867	179.799350	0.36095728545562500000	0.99113097519138400000

Table 4: Predicted orthometric heights of station outside the observation Stations

Station no.	Observed orthometric height(m)	Predicted orthometric height(m)	Difference (E)
DOS07S	7.516784105	7.516782253	0.0000018518
SGIS006	4.315235000	4.315234864	0.0000001359
ED015	5.830372000	5.830371354	0.0000006458
SGYM017	5.839751000	5.839750620	0.0000003804
SGJA015	6.928477341	6.928476306	0.0000010355
SGDL014	2.937411803	2.937411999	-0.0000001959
PGS9	7.020878584	7.020877325	0.0000012592
XST347AZ	5.351150088	5.351148918	0.0000011698
MEGA11	5.049271000	5.049270720	0.0000002798
ED009	5.749291000	5.749290622	0.0000003780
ED006	7.548386694	7.548385815	0.0000008787
DSG07	1.641999968	1.642000427	-0.0000004592
XST347	4.701000000	4.700996942	0.0000030578
SGEG02	1.815455359	1.815457168	-0.0000018086
USL01	8.479188863	8.479187457	0.0000014055
SGEN012	4.972066000	4.972065753	0.0000002474
SGUG007	3.807963000	3.807962660	0.0000003401
DSG003	2.129748152	2.129748424	-0.0000002721
ED014	5.991619009	5.991618057	0.0000009518
CR5	3.521918000	3.521918526	-0.0000005256
CR8	4.191252000	4.191254523	-0.0000025231
CBLM3	4.633840277	4.633841102	-0.0000008249
SGRN008	3.450022000	3.450022126	-0.0000001260
YTT28-186	6.265957000	6.265955970	0.0000010303
SGCP016	7.414177278	7.414176179	0.0000010989
DOS04S	5.575347000	5.575346830	0.0000001700
PGD81-1	6.604817993	6.604817556	0.0000004372
CR3	3.945358000	3.945357817	0.0000001826
DOS03S	6.626035000	6.626033890	0.0000011103
SGEG003	7.525433165	7.525431808	0.0000013565

Table 5: Predicted orthometric heights of station outside the observation Stations

Station no.	Observed orthometric height(m)	Predicted orthometric height(m)	Difference (E)
SGEG01	2.530312838	2.495110443	0.035202
ED011	3.382621938	3.159618155	0.223004
SD14S	3.227390000	3.060610187	0.166780
SGEN010	5.563834000	5.356577680	0.207256
MEGA05	1.824800000	1.928847432	-0.104047
MEGA10	6.222928000	6.420759143	-0.197831
SGEG009	5.822543000	5.895826042	-0.073283
SGLF005	3.202049358	3.371133454	-0.169084
CR6	3.396077000	3.647761153	-0.251684
GMS05	8.247086872	8.396472631	-0.149386
SGEN011	4.977269000	4.771819954	0.205449
MEGA04	5.872013000	5.805749766	0.066263
DOS14S	5.883506000	6.014733596	-0.131228
ED013	5.075428000	5.007711625	0.067716

Table 6: Computed and Table Statistics at α=0.05

Degree of freedom	16
Table Statistic (Upper limit)	28.845
Computed Statistic	14.959
Table Statistic (Lower limit)	6.908

4. Analysis of Results

The results in table 4 show that the predicted orthometric heights of observation stations compare favorably well with the observed orthometric heights of those stations. Also, in table 5, the predicted orthometric heights obtained outside the observation stations seem to have satisfactory level of reliability. This is because the difference between the predicted height and the actual height of each station appears to be insignificant. Furthermore, statistical investigation was carried out to test the level of reliability of the predicted orthometric heights obtained for stations outside the observation stations. This was to show whether or not the procedure used had introduced distortions in the predicted heights. In order words, $E^T C_{hh}^{-1} E$ was statistically examined to know whether it falls within the specific confidence limit or not. This was achieved by means of Chi Squares (χ^2) test. That is, we tested the hypothesis:

Null hypothesis: H$_0$: $E^T C_{hh}^{-1} E = \sigma_0^2$ ($E^T C_{hh}^{-1} E$ is within the confidence limit)

Alternative hypothesis: H$_1$: $E^T C_{hh}^{-1} E \neq \sigma_0^2$ ($E^T C_{hh}^{-1} E$ is outside the confidence limit)

Where $E^T C_{hh}^{-1} E / \sigma_0^2$ is the computed statistics (χ^2).

This is a two tailed test where the null hypothesis is rejected if the computed statistic is outside the confidence limits. The confidence limits are the upper limit and the lower limit of the table statistics. They are obtained in statistical table as $\chi^2_{1-\alpha/2,df}$ for upper limit and $\chi^2_{\alpha/2,df}$ for lower limit, where α is the level of significance and *df* is the degree of freedom (i.e. number of observation minus the number of predicted orthometric heights (*u*)). From table 6, it can be inferred that the value of computed statistics falls within the confidence limits. This suggests that the null hypothesis ($E^T C_{hh}^{-1} E$) is within the confidence limit and should not be rejected. Therefore, one can rely on the assumption that the least squares collocation technique used for the prediction of the orthometric heights has not introduced significant distortion in the predicted heights. Also, the values of E were examined using statistical t-distribution to show whether or not they fall within the tolerant error limit (e) for the predicted heights. The tolerance error limit (e) is define by equations 13 and 14 (Ayeni, 2001).

$$e = \pm \sigma t_{u-1,1-\alpha/2} / u^{1/2} \tag{13}$$

$$\sigma = (E^T C_{hh}^{-1} E / (u-1))^{1/2} \tag{14}$$

From the statistical table, $t_{u-1,1-\alpha/2}$ = 2.131. Also, the computed value of σ = 0.999. Hence, the value of tolerant error limit (e) = $\pm$ 0.532. Therefore, from table 5, it can be inferred that all the values of *E* fall within the tolerant error limit (e) thereby confirming the high level of reliability of the predicted orthometric heights.

5. Conclusions and Recommendations

In this paper, prediction of orthometric height using least squares collocation has been attempted. The good search and reliable choice of covariance parameters for the design of covariance function had helped in ensuring that a high level of reliability of the predicted orthometric heights was achieved. The predicted heights have been found to be satisfactory at the significance level of 0.05. Therefore, prediction of orthometric heights of stations is recommended to improve the density the of available orthometric heights in a survey project area.

References

Ayeni O. O. (2001): Statistical Adjustment and Analysis of Data. Lecture note series, Department of Surveying and Geoinformatics, University of Lagos, Nigeria.

Ezeigbo C. U. (1988): Least squares collocation method of geoid and datum determination for Nigeria. Ph. D. Thesis, Department of Surveying, University of Lagos, Nigeria.

Gerdan, G.P. (1991): Rural cadastral surveying with GPS. Journal of Australia Surveyor, Vol. 36, No. 3 pp 42-46.

Idowu, T. O. (2007): On the statistical distribution of Global Positioning (GPS) data. Journal of Environmental Sciences, Vol. 11(1) pp 140-144.

Idowu, T. O. (2006): Prediction of gravity anomalies for geophysical exploration. FUTY Journal of the Environment, Vol. 1, No. 1, pp 44-55.

Idowu, T. O. (2005): Determination and Utilization of optimum residual gravity anomalies for mineral exploration. Ph. D Thesis, Department of Surveying and Geoinformatics, University of Lagos, Nigeria

Krakiwsky E. J. (1975): Recent Advances in least squares: Lecture Note Series. University of News-Brunswich, Canada.

Krarup, T. (1970): The method of least squares collocation. Studio Geophysica.

Moka, E. C. (2011): Requirements for the determination of orthometric heights from GPS-determined heights. In Fajemirokun, F. A. (ed): “Contemporary Issues in Surveying and Geoinformatics”. ISBN 97 8-915-670-2. BPrint, Ikeja, Lagos.

Moritz H. (1972): Advanced least squares method. Report No. 175, Department of Geodetic Science, The Ohio State University.

Moritz H. (1978): Least squares collocation. Journal of American Geophysical Union, **16** 201-5.

Rapp R. H. (1986): Global Geopotential Solutions. Lecture Notes in Earth Sciences. Department of Geodetic Sciences and Surveying, Ohio State University.

Schwarz K. P. (1976): Least Squares Collocation for large systems. Boll. Geod. Scl. Affini. 35 : 309-324.

Tiberius, C. C. J. M. and Borre, K. (2000): Are GPS data normally distributed? Proceeding of International Association of Geodesy Symposia. Vol. 121, pp 243-247.