# Scattering By Randomly placed Perfectly Electromagnetic Conducting Half Plane

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## Abstract

*An analytic theory for the electromagnetic scattering from a randomly placed perfectly electromagnetic conducting (PEMC) half plane is developed, using the duality transformation which was introduced by Lindell and Sihvola. The theory allows for the occurrence of cross-polarized fields in the scattered wave, a feature which does not exist in standard scattering theory. This is why the medium is named as PEMC. PEMC medium can be transformed to PEC or PMC media. As an application, plane wave reflection from a planar interface of air and PEMC medium is studied. PEC and PMC are the limiting cases, where there is no cross-polarized component*

## 1 Introduction

The problem we are considering, i.e., scattering from half plane, strip or grating are very well known in the field of electromagnetics [1]. Our aim is not to resolve these problems but introduce few random parameters in these planner boundaries. For perfectly electric conducting cases, a complete solution exists in literature, based on the following equations and conditions, first time the half plane problem was solved exactly by A. Sommerfeld in 1896 [2], Later, the same solution was obtained by several different methods, [3, 4] and to study the effects of the stochastic nature of these boundaries on the scattered field. Before we examine the random boundaries, i.e., scatterers with random parameters is instructive to examine the behavior of randomly placed half plane, because in two dimensional planner perfectly conducting boundaries, with sharp edges. In this paper, the solution for the following average scattered field for pec case has been tranformed to randomly placed pemc half plane.

## 2 Formulation of The Problem

Consider a perfectly conducting plane is located at $(x > 0, y = 0)$ Fig.(1) and illuminated by an incident plane wave. It is convenient to resolve the fields into two modes: $E$-wave and $H$-wave, and solve the problem associated with each mode separately.

For convenience, a scalar function $U$ is introduced such that

$$U(x, y) = \{E_z, for E - wave H_z, for H - wave. \tag{1}$$

The plane wave is specified for all $(r, \phi)$,

$$U(r, \phi) = e^{ikr\cos(\phi^i - \phi)} \tag{2}$$

$$U^T = 0, \tag{3}$$

$H$-wave for $x > 0$ and $y = 0$:

$$\frac{\partial U^T}{\partial y} = 0, \tag{4}$$

The edge condition for $U^T$,

$$\frac{\partial U^T}{\partial x}, \frac{\partial U^T}{\partial y} = O(r^{-\frac{1}{2}}), \tag{5}$$

as $r \to 0$ Finally, thhe radiation conditions for the scattered field $U$ as $r$ approaches to infinity.

The total scattered field in far zone is given below.

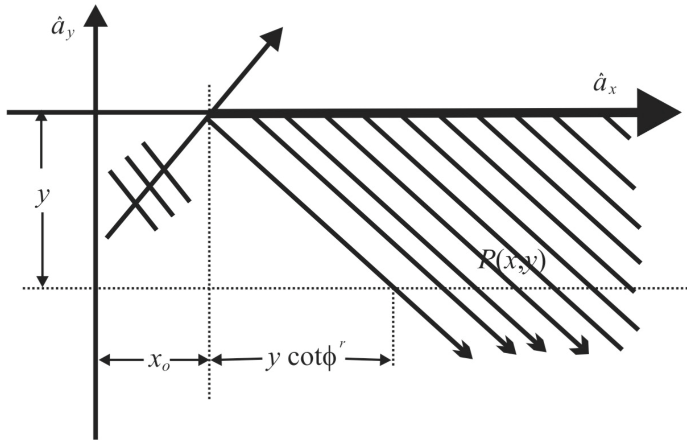


**Figure 1: scattering by randomly placed pec half plane**

$$U(r,\phi) = Re^{ikr\cos(\phi^r-\phi)} + C\frac{e^{ikr}}{\sqrt{kr}}U^i(r=0) \tag{6}$$

where $C = [csc\frac{(\phi^i-\phi)}{2} - csc\frac{(\phi^r+\phi)}{2}]\frac{e^{i\frac{\pi}{4}}}{2\sqrt{2\pi}}$

Consider the geometry as sketched in Fig.(1), where a perfectly conducting half plane is located at $x > x_o$ and $y = 0$. Random placement of this half plane is due to the edge location or $x_o$ be random variable with certain probability density function. The main interes is in the scattering of plane wave through this geometry. Due to random $x_o$ the scattered field is also random, and in the statistics of this field.

Consider a simple displaced half plane field; later the statistics of this field is calculate . The form of the total field will be similar to basic equation with little modification,

$$U^T(r,\phi) = F(\xi'^i)U^i(r',\phi) + F(\xi'^r)U^r(r',\phi) \tag{7}$$

where $r' = \sqrt{(x-x_o)^2+y^2}$, only the far zone solution is of interest. The scattered field could be separated in two parts Geometrical Optic Field and Keller,s Diffracted Field, i.e.,

$$U(r,\phi) = U^g(r,\phi) + U^d(r,\phi) \tag{8}$$

The Geometric Optic Field is given by

$$U^g(r,\phi) = f(r,\phi)U^r(r,\phi) \tag{9}$$

where $f(r,\phi) or f(x,y)$ is either $0$ or $1$. The function $f(x,y) = 1$ when point is in the shaded region, as shown in the 1 and $f(x,y) = 0$ when point is outside the region. In this case the function is replaced by unit step function,

$$f(x,y) = u(\tilde{x} - x_o) \tag{10}$$

where $u(x)$ is defined as

$$u(y) = \{1, if \ \ y > 0$$

and $\tilde{x} = x - ycot\phi^r$. The second part of the scattered field which is due to the edge diffraction will be,

$$U^d(x,y) = G(k\sqrt{(x-x_o)^2+y^2})(\chi^i + \mathbb{R}\chi^r)U^i(r=x_o) + O(k^{\frac{-3}{2}}) \tag{11}$$

For $E$-wave the $z$-component of the scattered field will become,

$$E_z^s(x,y) = E^r(x,y) + E^d(x,y) \tag{12}$$

where $E^r(x,y) = U^g(x,y)$ and $E^d(x,y) = U^d(x,y)$, for $R = -1$, this implies

$$E^r(x,y) = u(\tilde{x} - x_o)U^r(x,y) \tag{13}$$

where $U^r(x,y) = Re^{ik\sqrt{x^2+y^2}}\cos(\phi^r - \phi)$

The diffracted part is given as,

$$E^d(x,y) = G(k\sqrt{(x-x_o)^2 + y^2})(\chi^i + R\chi^r)U^i(r = x_o) + O(k^{\frac{-3}{2}}) = \frac{1}{2}\sqrt{\frac{i}{2\pi}}(\chi^i + R\chi^r)e^{ikx_0\cos(\phi^i)}\frac{e^{ik\sqrt{(x-x_o)^2+y^2}}}{k\sqrt{(x-x_o)^2+y^2}} \tag{14}$$

When $r >> x_o$, far from edge , it can be approximated

$$\frac{e^{ikr\sqrt{(x-x_o)^2+y^2}}}{\sqrt{k\sqrt{(x-x_o)^2+y^2}}} \cong \frac{e^{ikr}}{\sqrt{kr}}e^{-ikx_o\cos\phi}\frac{e^{i\sqrt{(x-x_o)^2+y^2}}}{k\sqrt{(x-x_o)^2+y^2}} \cong \frac{e^{ikr}}{\sqrt{kr}}e^{-ikx_o\cos(\phi^i)} \tag{15}$$

where again we can not neglect exponential term otherwise there will be phase error.

$$E^d(x,y) \cong Ce^{ika\,x_o}\frac{e^{ikr}}{\sqrt{kr}} \tag{16}$$

where $a = \cos(\phi^i) - \cos(\phi^r)$, and

$$C = \frac{1}{2}\sqrt{\frac{i}{2\pi}}(\chi^i + R\chi^r), \qquad 0 < \phi^i < \pi$$

The average scattered field is the sum of average reflected and diffracted field i.e.,

$$< E_z^s > = < E^r > + < E^d > \tag{17}$$

The variance of scattered field is given by,

$$var(E_z^s) = var(E^r) + var(E^d) + cov(E^rE^d) + cov(E^dE^r) \tag{18}$$

where $cov(XY)$ is the covariance of $X$ and $Y$ defined by the equation,

$$cov(XY) = < (X - < X >)(Y - < Y >)^* > \tag{19}$$

n order to calculate the second order statistics of the scattered field, i.e., its average and variance, we need to calculate the average reflected field, average diffracted field, their variances and the correlation between reflected and diffracted fields. In following section we calculate them successively.

## *3 Average Reflected Field and its Variance*

Obviously, the reflected field at point $P(x,y)$ is due to reflection from perfectly conducting plane. If trace back from $P(x,y)$ on the $xz$-plane along the reflected angle, suppose $P'(x,y)$ from where the reflection is obtained. Since $x_o$ is random or exact location of edge is unknown, there is a chance or probability that point $P'(x,y)$ lies on plane or does not lie on the plane. So average reflected field will be the reflected $U^r(x,y)$ multiplied by the probability that $P'(x,y)$ lies on plane.Taking average on both sides of this equation,

$$< E^r(x,y) >=< u(x-x_o) > U^r(x,y) \tag{20}$$

while

$$< u(x-x_o) >= \int_{-\infty}^{\infty} u(x-x_o) P_{x_o} dx_o = \int_{-\infty}^{\infty} P_{x_o} dx_o \tag{21}$$

Variance of this field can be calculated as,

$$var(E^r) =< |E^r|^2 > -| < E^r > |^2 \tag{22}$$

Let us take exponentially distributed $x_0$, for simplicity, whose probability density function is defined as

$$< u(\tilde{x}-x_o) >= \lambda \int_0^{\tilde{x}} e^{-\lambda(x_o)} dx_o = 1 - e^{-\lambda(\tilde{x})} \tag{23}$$

$$var(E^r) = e^{-\lambda(\tilde{x})}(1 - e^{-\lambda(\tilde{x})}) \tag{24}$$

It can be seen that average reflected is still a plane wave travelling in the $\phi^r$ direction, whose strength increases exponentially to unity(for $\phi > \phi^r$) as moving from $x > y\cot\phi^r$ towards positive axis, in any plane parallel to $xz$-plane at depth y. The strength of average reflected field and its variance 1 at any plane parallel to$xz$-plane at depth y for $\hat{x} \geq 0$.

## 4 Average Diffracted Field and its Variance

The approximate diffracted field, due to edge effect, which is same as the field radiated by edge source, placed at $x^o$, in the far zone. Since its location is random, the average field radiated by this source will be,

$$< E^d(x,y) >= C < e^{ika\,x_o} > \frac{e^{ikr}}{\sqrt{kr}} \tag{25}$$

where $C = \frac{1}{2}\sqrt{\frac{i}{2\pi}}(\chi^i + R\chi^r)$

and the average term in above equation is given by

$$< e^{ika\,x_o} >= \frac{1}{1-ik<x_o>}(\cos\phi^i - \cos\phi) \tag{26}$$

where $< x_o >$ is the average value of $x_o$. The average diffracted field will become

$$< E^d(x,y) >= C\frac{e^{ikr}}{\sqrt{kr}}\frac{1}{1-ik<x_o>(cos\,\phi^i - \cos\phi)} \tag{27}$$

$$\langle E^d(x,y)\rangle = C\frac{e^i kr}{\sqrt{kr}}\frac{1}{1-ik\langle x_o\rangle(\cos\phi^i - \cos\phi)} \tag{28}$$

The variance of this field is defined as

$$var(E^d) =< |E^d|^2 > +| < E^d > |^2 \tag{29}$$

$$var(E^d) = \frac{k<x_o>(cos\,\phi^i - \cos\phi)^2}{1+(k<x_o>(cos\,\phi^i - \cos\phi))^2} \times \frac{|C|^2}{kr} \tag{30}$$

## 5 Covariance between Reflected and Diffracted Fields

The covariance between two random variables is defined as

$$cov(E^r E^d) =< E^r E^{d^*} > -< E^r >< E^d >^* \tag{31}$$

where $< E^r E^{d^*} >$ is the cross correlation between reflected and diffracted fields and its expectation be written as

$$\langle E^r E^{d^*}\rangle = RC^* e^{ikr\cos(\phi^r - \phi)}\frac{e^{-ikr}}{\sqrt{kr}}\langle e^{-ika\,x_o} u(\tilde{x}-x_o)\rangle \tag{32}$$

where

$$\langle e^{-ika\,x_o} u(\tilde{x}-x_o)\rangle = \frac{1-e^{(-\lambda-ika)\tilde{x}}}{(1+ika<x_o>)} \quad (33)$$

$$< E^r E^{d^*} >= RC^* e^{ikr\cos(\phi^r-\phi)} \frac{e^{-ikr}}{\sqrt{kr}} \frac{1-e^{(-\lambda-ika)\tilde{x}}}{1+ika<x_o>} \quad (34)$$

and

$$cov(E^r E^{d*}) = e^{-\lambda\tilde{x}}(1-e^{-ika\,\tilde{x}}) Re^{ikr\cos(\phi^r-\phi)} < E^d >^* u(\tilde{x}) \quad (35)$$

$$var(E_z^s) = var(E^r) + var(E^d) + 2\Re(cov(E^r E^d)) \quad (36)$$

Now putting the values, we obtain the averaged scattered field as,

$$\langle E_z^s\rangle = (1-e^{-\lambda\tilde{x}})u(\tilde{x})Re^{ik\sqrt{x^2+y^2}}\cos(\phi^i-\phi) + C\frac{e^{ikr}}{\sqrt{kr}}\frac{1}{1-ik<x_o>(\cos\phi^i-\cos\phi)} \quad (37)$$

The expression for variance of scattered field $E_z^s$ field can be written as,

$$var(E_z^s) =$$
$$e^{-\lambda\tilde{x}}(1-e^{-\lambda\tilde{x}}) + \frac{(k<x_o>a)^2}{1+(k<x_o>a)^2} \times \frac{|C|^2}{kr} + 2\Re(C^* e^{-\lambda\tilde{x}}(1-e^{-\lambda\tilde{x}})Re^{ikr\cos(\phi^r-\phi)}\frac{e^{-ikr}}{\sqrt{kr}}\frac{1}{1+ika<x_o>}u(\tilde{x}) \quad (38)$$

$$< E^d(x,y) >= C < e^{ika\,x_o} > \frac{e^{ikr}}{\sqrt{kr}} \quad (39)$$

where $C = \frac{1}{2}\sqrt{\frac{i}{2\pi}}(\chi^i + R\chi^r)$

and the average term in above equation is given by

$$< e^{ika\,x_o} >= \frac{1}{1-ik<x_o>(\cos\phi^i-\cos\phi)} \quad (40)$$

where $< x_o >$ is the average value of $x_o$. The average diffracted field will become

$$< E^d(x,y) >= C(\frac{e^{ikr}}{\sqrt{kr}})\frac{1}{1-\imath k<x_o>(\cos\phi^i-\cos\phi)} \quad (41)$$

The variance of this field is defined as

$$var(E^d) =< |E^d|^2 > +| < E^d > |^2 \quad (42)$$

The above average scattered field can be transformed from perfectly electric conducting case to perfectly electromagnetic conducting case by the following theory. The Concept of PEMC introduced by Lindell and Sihvola [3, 4] is a generalization of both PEC and PMC. An analytic theory for the electromagnetic scattering from a PEMC plane where a line source has been placed randomly, is developed. The PEMC medium characterized by a single scalar parameter $M$, which is the admittance of the surface interface, where $M = 0$ reduces the PMC case and the limit $M \rightarrow \pm\infty$ corresponds to the perfect electric conductor (PEC) case. The theory allows for the occurrence of cross-polarized fields in the scattered wave in the scattered wave, a feature which does not exist in standard scattering theory. This means that PEC and PMC are the limiting cases, for which there is no cross-polarized component. Because the PEMC medium does not allow electromagnetic energy to enter, an interface of such a medium behaves as an ideal boundary to the electromagnetic field. At the surface of a PEMC media, the boundary conditions between PEMC medium and air with unit normal vector $n$, are of the more general form. Because tangential components of the $E$ and $H$ fields are continuous at any interface of two media, one of the boundary conditions for the medium in the air side is $n \times (H + ME) = 0$, because a similar term vanishes in the PEMC-medium side.

The other condition is based on the continuity of the normal component of the $D$ and $B$ fields which gives another boundary condition as $n.(D - MB) = 0$.

Here, PEC boundary may be defined by the conditions

$$n \times E = 0, \quad n.B = 0 \tag{43}$$

While PMC boundary may be defined by the boundary conditions

$$n \times H = 0, \quad n.D = 0 \tag{44}$$

where $M$ denotes the admittance of the boundary which is characterizes the PEMC. For $M = 0$, the PMC case is retrieved, while the limit $M \to \pm\infty$ corresponds to the PEC case. Possibilities for the realization of a PEMC boundary have also been studied [5].

It has been observed theoretically that a PEMC material acts as a perfect reflector of electromagnetic waves, but differs from the PEC andthe PMC in that the reflected wave has a cross-polarized component.

The duality transformations of perfectly electric condutor (PEC) to PEMC have been studied by many researchers [3, 4, 5, 6, 7, 8, 9]. Here we present an analytic scattering theory for a PEMC step, which is a generalization of the classical scattering theory.

Applying a duality transformation which is known to transform a set of fields and sources to another set and the medium to another one. In its most general form, the duality transformation can be defined as a linear relation between the electromagnetic fields. The effect of the duality transformation can be written by the following special choice of transformation parameters:

$$\begin{pmatrix} E_d \\ H_d \end{pmatrix} = \begin{pmatrix} M\eta_0 & \eta_0 \\ \frac{-1}{\eta_0} & M\eta_0 \end{pmatrix} \begin{pmatrix} E \\ H \end{pmatrix} \tag{45}$$

has the property of transforming PEMC to PEC, while

$$\begin{pmatrix} E \\ H \end{pmatrix} = \frac{1}{(M\eta_0)^2+1} \begin{pmatrix} M\eta_0 & -\eta_0 \\ \frac{1}{\eta_0} & M\eta_0 \end{pmatrix} \begin{pmatrix} E_d \\ H_d \end{pmatrix} \tag{46}$$

has the property of transforming PEC to PEMC [4].

Following the above relations [3], the transformed equations become as

$$E^r = -\frac{1}{M^2\eta_0^2+1}\left[(-1 + M^2\eta_0^2)E^i + 2M\eta_0 u_z \times E^i\right] \tag{47}$$

$$E_{sd} = -(M\eta_0 E_s + \eta_0 H_s) \tag{48}$$

$$H_{sd} = -\frac{1}{\eta_0}E_s + M\eta_0 H_s \tag{49}$$

$$E_s = \frac{1}{(M\eta_0)^2+1}\left[M\eta_0 E_{sd} - \eta_0 H_{sd}\right] \tag{50}$$

$$E_s = \frac{1}{(M\eta_0)^2+1}\left[((M\eta_0)^2 - 1)E_s - 2M\eta_0^2 H_s\right] \tag{51}$$

$$E_s = \frac{1}{(M\eta_0)^2+1}\left[((M\eta_0)^2 - 1)E_s - 2M\eta_0 E_s\right] \tag{52}$$

Where $E_s$, $H_s$ are transformed pemc average fields and $E_{sd}$, $H_{sd}$ are average scattered elecric and magnetic fields respectively.

This means that, for a linearly polarized incident field $E^i$, the reflected field from a such a boundary has a both co-polarized component, while $u_z \times E^i$ is a cross-polarized component, in the general case. For the PMC and PEC special cases ($M = 0$ and $M = \pm\infty$ respectively), the cross-polarized component vanishes.

For the special PEMC case $M = \frac{1}{\eta_0}$, such that

$$(E^r = -u_z \times E^i) \tag{53}$$

which means that the reflected field appears totally cross-polarized. It is obvious theoretically that a PEMC material acts as a perfect reflector of electromagnetic waves, but differs from the PEC $(E^r + E^i) = 0$ and $H^r = H^i$) and PMC ( $E^r = E^i$ and $H^r + H^i = 0$) in that the reflected wave has a cross-polarized component.

### 6 Concluding remarks

In this work, a plane wave scattering by randomly placed perfectly electromagnetic conducting half plane is studied. The theory provides explicit analytical formulas for the electric and magnetic field. An other formulla has been derived for the relative contributions to the scattered fields of the co-polarized and the crosspolarized fields depend on parameter $M$. The cross-polarized scattered fields vanish in the PEC and PMC cases, and are maximal for $M = \pm 1$. In the general case,the reflected wave has both a co-polarized and a cross-polarized component. The above transformed solution presents an analytical theory for the scattering by randomly placed perfectly electromagnetic conducting half plane. It is clear from the above discussion that for $M \to \infty$ and $M \to 0$ correspond to the PEC and PMC respectively. Moreover, for $M = \pm 1$ the medium reduces to PEMC.

## 7 Acknowledgements

The author would like to acknowledge the Department Of Electronics Quaid-i-azam University Islamabad Pakistan for providing research environment and moral support.

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