Nonoscillation and Oscillation for a Class of Neutral Dynamic Equations with Positive and Negative Coefficients on Time Scales

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Abstract

In this paper, we study the existence of oscillatory and nonoscillatory solutions of neutral dynamic equations

$$(x(t)-cx(t-r))^{\Delta} \pm (P(t)x(t-\theta)-Q(t)x(t-\delta))=0$$

where $c>0, r>0, \theta>\delta\geq 0$ *are constants,* $P,Q\in C_{rd}(T,R^{+})$ *.We obtain some sufficient and necessary conditions for the existence of bounded and unbounded positive solutions, as well as some sufficient conditions for the existence of bounded and unbounded oscillatory solutions.*

Keywords: Nonoscillation, Oscillation, Time scales, Neutral

1. ITRODUCTION

In this paper, we consider the following neutral differential equation with positive and negative coefficients on time scales

$$(x(t)-cx(t-r))^{\Delta}+P(t)x(t-\theta)-Q(t)x(t-\delta)=0 \qquad (1.1)$$

and

$$(x(t)-cx(t-r))^{\Delta}=P(t)x(t-\theta)-Q(t)x(t-\delta) \qquad (1.2)$$

where $c>0, r>0, \theta>\delta\geq 0$ are constants, $P,Q\in C_{rd}(T,R^{+})$.

$$(x(t)-cx(t-r))'+P(t)x(t-\theta)-Q(t)x(t-\delta)=0 \qquad (1.3)$$

and

$$(x(t)-cx(t-r))'=P(t)x(t-\theta)-Q(t)x(t-\delta) \qquad (1.4)$$

Eq.(1.3) and Eq.(1.4) have been investigated by Graef and Yang and Zhang [2,6],Yu [9],Yu and Wang [8] and Lalli and Zhang [7]. However, the research on the existence of positive solutions and oscillatory solutions of (1.1) and (1.2) are scarce in the literature.

In Section 2, we obtain conditions for the existence of both bounded positive solutions and bounded oscillatory solutions for Eq.(1.1) with $c=1$.

In Section 3, we obtain conditions for the existence of unbounded positive solutions for Eq.(1.1) with $c=1$.

In Section 4, we obtain conditions for the existence of both bounded positive solutions and bounded oscillatory solutions for Eq.(1.1) with $c\in(0,1)$.

In Section 5, we obtain conditions for the existence of both bounded positive solutions and bounded oscillatory solutions for Eq.(1.2).

In Section 6, we consider Eq.(1.1) and Eq.(1.2) in the case $c>1$.The following hypothesis will be adopted throughout in this paper:

(H1) $r>0, \theta>\delta\geq 0$ are constants,

(H2) $P,Q\in C_{rd}(T,R^{+})$,

(H3) $\overline{P}(t)=P(t)-Q(t-\theta+\delta)\geq 0$.

The following lemma is taken from Zhang and Yu [10].

Lemma 1.1. Suppose that $f\in C_{rd}([t_0,\infty),R^{+})$ and $r>0$. Then

$$\sum_{j=0}^{\infty}\int_{t_0+jr}^{\infty} f(s)\Delta s<\infty$$

is equivalent to

$$\int_{t_0}^{\infty} tf(t)\Delta t < \infty .$$

2 Bounded solutions of Eq.(1.1) with c = 1.

In this section, we consider the equation

$$(x(t)-x(t-r))^{\Delta} + P(t)x(t-\theta) - Q(t)x(t-\delta) = 0 \qquad (2.1)$$

Theorem 2.1. Assume (H1),(H2),(H3) and the following conditions hold.

(H4) $\int^{\infty} s\overline{P}(s)\Delta s < \infty,$

(H5) $\int^{\infty} Q(s)\Delta s < \infty.$

Then Eq.(2.1) has a bounded positive solution, and for any continuous periodic oscillatory function $\omega(t)$ with period r, has a bounded oscillatory solution $x(t)$ such that

$$x(t) = \omega(t) + R(t) \qquad (2.2)$$

where $R(t)$ is a rd-continuous real function with $|R(t)| < \alpha M$, $M = \min\{\max \omega(t), \max(-\omega(t))\}$ $\alpha \in (0,1)$ for $t > t_1, t_1$ is a sufficiently large number.

To prove the above theorem, we need to establish the following lemmas.

Lemma 2.1 Suppose the assumptions of Theorem 2.1 hold. Then the equations

$$(x(t)-x(t-r))^{\Delta} + P(t)(x(t-\theta)+2\overline{M}+\omega(t-\theta)) - Q(t)(x(t-\delta)+2\overline{M}+\omega(t-\delta)) = 0 \quad (2.3)$$

and

$$(x(t)-x(t-r))^{\Delta} + P(t)(x(t-\theta)+2\overline{M}) - Q(t)x(t-\delta)+2\overline{M}) = 0 \qquad (2.4)$$

have bounded positive solutions $u_1(t)$ and $u(t)$ respectively, such that

$$|u(t)| \le \frac{\alpha}{2}M, |u_1(t)| \le \frac{\alpha}{2}M$$

for $t \ge t_1$,where $\overline{M} = \max|\omega(t)|$, t_1 is sufficiently large.

Proof. The proof about Eq.(2.3) is quite similar to that about Eq.(2.4), we give only the proof about Eq.(2.4). Choose t_1 sufficiently large such that

$$\sum_{i=0}^{\infty} \int_{t_1+ir}^{\infty} \overline{P}(s)\Delta s + n\int_{t_1-\theta}^{\infty} Q(s)\Delta s < \frac{\alpha M}{16\overline{M}} \qquad (2.5)$$

where $n = \left[\frac{\theta-\delta}{r}\right] + 2$, Set

$$H(t) = \begin{cases} 4\overline{M}\int_t^{\infty} \overline{P}(s)\Delta s + 4\overline{M}\int_{t-\theta+\delta}^{t} Q(s)\Delta s, & t \ge t_1 \\ (t-t_1+r)H(t_1)/r & t_1 - r \le t \le t_1 \\ 0 & t \le t_1 - r. \end{cases}$$

Clearly, $H \in C_{rd}(T, R^+)$. Define $y(t) = \sum_{i=0}^{\infty} H(t-ir), t > t_1$.

It is obvious that $y \in C_{rd}([t_1,\infty), R^+)$ with $y(t) - y(t-r) = H(t)$, $0 < y(t) < \frac{\alpha}{4}M < \overline{M}$, $t \ge t_1$.

Define a set X by

$$X = \{x \in C_{rd}([t_1,\infty),R) | 0 \le x(t) \le y(t), t \ge t_1\}$$

and an operator S on X by

$$(Sx)(t) = \begin{cases} x(t-r) + \int_{t-\theta+\delta}^{t} Q(s)(x(s-\delta)+2\overline{M})\Delta s + \int_{t}^{\infty} \overline{P}(s)(x(s-\delta)+2\overline{M})\Delta s & t \ge t_1 + m \\ (Sx)(t_1+m) & t \in [t_1, t_1+m] \end{cases}$$ where

$m = \max\{\theta, r\}$.

Clearly

$$(Sx)(t) \le y(t-r) + H(t) = y(t), t \ge t_1 + m,$$

for any $x \in X$ .i.e., $SX \subset X$.

Define a sequence of functions $\{x_k(t)\}_{k=0}^{\infty}$ as follows:

$$x_0(t) = y(t), t \ge t_1,$$
$$x_k(t) = (Sx_{k-1})(t), t \ge t_1, k = 1,2,\cdots.$$

By induction we can prove that

$$0 < x_k(t) \le x_{k-1}(t) \le y(t), t \ge t_1, k = 1,2,\cdots.$$

Then there exists a function $u \in X$ such that $\lim_{k\to\infty} x_k(t) = u(t), t \ge t_1$. It is obvious that $u(t) > 0$ on $[t_1,\infty)$. By Lebesgue's dominated convergence theorem we have

$$u(t) = u(t-r) + \int_{t-\theta+\delta}^{t} Q(s)(u(s-\delta)+2\overline{M})\Delta s + \int_{t}^{\infty} \overline{P}(s)(u(s-\theta)+2\overline{M})\Delta s, t \ge t_1 + m.$$

Then we get

$$(u(t)-u(t-r))^{\Delta} = Q(t)(u(t-\delta)+2\overline{M}) - P(t)(u(t-\theta)+2\overline{M}).$$

i.e., $u(t)$ is a bounded positive solution of Eq.(2.4) with $0 < u(t) \le \frac{\alpha}{4}M$. The proof is complete.

Proof of Theorem 2.1. Let

$$U(t) = 2\overline{M} + u(t), U_1(t) = 2\overline{M} + \omega(t) + u_1(t).$$

where $u(t), u_1(t)$ are defined by Lemma 2.1. It is easy to see that $U(t)$ and $U_1(t)$ are both bounded positive solutions of Eq.(2.1). Because Eq.(2.1) is linear,

$$x(t) = U_1(t) - U(t) = \omega(t) + (u_1(t) - u(t)), t \ge t_1$$

is also a solution of Eq.(2.1). It is easy to see that x(t) is oscillatory and satisfies (2.2).The proof is complete.

Example 2.1. If $T = R$, consider the neutral differential equation

$$(x(t)-x(t-1))' + P_1(t)x(t-1) - Q_1(t)x(t) = 0 \qquad (2.6)$$

Where $P_1(t) = \frac{6}{t^2(t-1)(t-2)}, Q_1(t) = \frac{4t-1}{t(t-1)^4(t+1)}$.

Clearly $\overline{P_1}(t) \equiv P_1(t) - Q_1(t-1) \ge 0$,

$$\int^{\infty} Q_1(s)ds < \infty \text{ and } \int^{\infty} s\overline{P_1}(s)ds < \infty.$$

By Theorem 2.1,Eq.(2.6) has a bounded positive solution. In fact, $x(t) = 1 - t^{-2}$ is such a solution of Eq.(2.6).

Example 2.2. If $T = R$, consider the neutral differential equation

$$(x(t) - x(t-2\pi))^{\Delta} + P_2(t)x(t - \frac{5}{2}\pi) - Q_2(t)x(t-\pi) = 0 \qquad (2.7)$$

Where $P_2(t) = \frac{4\pi(t-\pi)}{t^2(t-2\pi)^2} \cdot \frac{(t-\frac{5}{2}\pi)^2}{(t-\frac{5}{2}\pi)^2 - 1}, Q_2(t) = 4\pi \frac{3t^2 - 6\pi t + 4\pi^2}{(t(t-2\pi))^3} \cdot \frac{(t-\pi)^2}{(t-\pi)^2 - 1}$.

Clearly $\overline{P}_2(t) \equiv P_2(t) - Q_2(t - \frac{3}{2}\pi) \geq 0$,

$$\int^{\infty} Q_2(s)ds < \infty \text{ and } \int^{\infty} s\overline{P}_2(s)ds < \infty.$$

By Theorem 2.1,Eq.(2.7) has a bounded oscillatory solution. In fact, $x(t) = (1 - t^{-2})\sin t$ is such a solution of Eq.(2.7).

The following result is about necessary conditions for the existence of bounded positive solutions of (2.1).

Theorem 2.2. Assume that (H1),(H2),(H3) and (H5) hold. If Eq.(2.1) has a bounded positive solution. Then (H4) holds.

Proof. Let x(t) be a bounded positive solution of Eq.(2.1). Then there exists $L > 0$ and $t_0 > 0$ such that $0 < x(t) < L$ on $[t_0, \infty)$. Set

$$y(t) = x(t) - x(t-r) - \int_{t-\theta+\delta}^{t} Q(s)x(s-\delta)\Delta s$$

Then

$$y^{\Delta}(t) = -\overline{P}(t)x(t-\theta) \leq 0, t \geq t_0 \qquad (2.8)$$

We claim that $y(t) > 0$ eventually. Assume the contrary that $y(t) < 0$ eventually.

Then there exist $t_1 > t_0$ and $\alpha > 0$ such that $y(t) \leq -\alpha$ on $[t_1, \infty)$. Then

$$x(t) \leq -\alpha + x(t-r) - \int_{t-\theta+\delta}^{t} Q(s)x(s-\delta)\Delta s$$

for $t \geq t_1$. By induction we have

$$x(t_1 + kr) \leq -k\alpha + x(t_1) + \sum_{i=1}^{k} \int_{t_1+ir-\theta+\delta}^{t_1+ir} Q(s)x(s-\delta)\Delta s \leq -k\alpha + x(t_1) + nL\int_{t_1-\theta}^{\infty} Q(s)\Delta s$$

where $n = \left[\frac{\theta-\delta}{r}\right] + 2$, $k = 1,2,\cdots$. Then $x(t_1 + kr) < 0$ for sufficiently large k, which is a contradiction. So we have

$$x(t) > x(t-r) + \int_{t-\theta+\delta}^{t} Q(s)x(s-\delta)\Delta s > x(t-r)$$

eventually. Then there exists $J > 0$ and $t_2 > t_1$ such that $x(t) > J$ x(t) > J on $[t_2, \infty)$. From (2.8), we see

$$y^{\Delta}(t) \leq -\overline{P}(t)J, t \geq t_3 = t_2 + \theta.$$

Integrating its both sides, we get

$$y(t) \geq J\int_{t}^{\infty} \overline{P}(s)\Delta s.$$

Then

$$x(t) \geq x(t-r) + \int_{t-\theta+\delta}^{t} Q(s)x(s-\delta)\Delta s + J\int_{t}^{\infty} \overline{P}(s)\Delta s \geq x(t-r) + J\int_{t}^{\infty} \overline{P}(s)\Delta s, t \geq t_3$$

Therefore,

$$L \geq x(t_3 + kr) \geq x(t_3) + J\sum_{i=1}^{k} \int_{t_3+ir}^{\infty} \overline{P}(s)\Delta s, \qquad (2.9)$$

for $k = 1,2,\cdots$. Letting $k \to \infty$ in (2.9), we get

$$\sum_{i=1}^{\infty} \int_{t_3+ir}^{\infty} \overline{P}(s)\Delta s < \infty,$$

which is equivalent to

$$\int^{\infty} s\overline{P}(s)ds < \infty.$$

The proof is complete.

Corollary 2.1. Assume that (H1),(H2),(H3) and (H5) hold. Eq.(2.1) has a bounded positive solution if and only if (H4) holds.

3 Unbounded solutions for Eq.(1.1) with c = 1

Definition 3.1. A positive solution x(t) of Eq.(2.1) is called an A-type solution, if it can be expressed in the form

$$x(t) = \alpha t + \beta t, \tag{3.1}$$

where $\alpha > 0$ is a constant, $\beta : [t_x, \infty) \to R$ is a bounded continuous function, $t_x > 0$.

Theorem 3.1. Assume (H1),(H2),(H3) and the following conditions hold.

(H6) $\int^{\infty} t^2 \overline{P}(t)dt < \infty$,

(H7) $\int^{\infty} tQ(t)dt < \infty$,

Then Eq.(2.1) has an A-type positive solution.

Proof. Choose t_1 sufficiently large such that

$$\sum_{i=0}^{\infty}\int_{t_1+ir}^{\infty} \overline{P}(t)(t+1)dt + n\int_{t_1-\theta}^{\infty} Q(t)(t+1)dt < 1,$$

where $n = \left[\dfrac{\theta-\delta}{r}\right] + 2$, Set

$$H(t) = \begin{cases} \int_t^{\infty} \overline{P}(s)(1+s)\Delta s + \int_{t-\theta+\delta}^{t} Q(s)(1+s)\Delta s, & t \ge t_1 \\ (t-t_1+r)H(t_1)/r & t_1 - r \le t \le t_1 \\ 0 & t \le t_1 - r. \end{cases}$$

Clearly, $H \in C_{rd}(T, R^+)$. Define $y(t) = \sum_{i=0}^{\infty} H(t-ir), t > t_1$.

It is obvious that $y \in C([t_1,\infty), R^+)$ with $y(t) - y(t-r) = H(t)$ and $0 < y(t) < 1,\ t \ge t_1$.

Define a set X by

$$X = \{x \in C_{rd}([t_1,\infty), R) | 0 \le x(t) \le y(t), t \ge t_1\}$$

and an operator S on X by

$$(Sx)(t) = \begin{cases} x(t-r) + \int_{t-\theta+\delta}^{t} Q(s)(x(s-\delta)+s-\delta)\Delta s + \int_t^{\infty} \overline{P}(s)(x(s-\theta)+s-\theta)\Delta s & t \ge t_1 + m \\ (Sx)(t_1+m) & t \in [t_1, t_1+m] \end{cases}$$

Where $m = \max\{\theta, r\}$.

Clearly, $SX \subset X$.

Define a sequence of functions $\{x_k(t)\}_{k=0}^{\infty}$ as follows:

$$x_0(t) = y(t), t \ge t_1,$$

$$x_k(t) = (Sx_{k-1})(t), t \ge t_1, k = 1,2,\cdots.$$

By induction we can prove that

$$0 < x_k(t) \le x_{k-1}(t) \le y(t), t \ge t_1, k = 1,2,\cdots.$$

Then there exists a function $u \in X$ such that $\lim_{k\to\infty} x_k(t) = u(t), t \ge t_1$. It is obvious that $u(t) > 0$ on $[t_1, \infty)$. By Lebesgue's dominated convergence theorem we have $u = Su$. It is easy to see that x(t) = t+u(t) is an A-type positive solution of Eq.(2.1). The proof is complete.

Similar to Theorem 2.2, we have

Theorem 3.2. Assume that (H1),(H2),(H3) and (H7) hold. If Eq.(2.1) has an A-type positive solution. Then (H6) holds.

Corollary 3.1. Assume that (H1),(H2),(H3) and (H7) hold.Eq.(2.1) has an A-type positive solution if and only if (H6) holds.

4 Bounded solutions of Eq.(1.1) with c ∈ (0, 1)

In this section, we consider the following equation

$$(x(t)-cx(t-r))^{\Delta}+(P(t)x(t-\theta)-Q(t)x(t-\delta))=0 \qquad (4.1)$$

where $c\in(0,1)$.

Theorem 4.1. Suppose that $c\in(0,1)$. (H1),(H2),(H3),(H5) and the following condition hold.

(H8) $\sum_{j=0}^{\infty}\int_{t_3+jr}^{\infty}c^{\frac{s-t_1-jr}{r}}\overline{P}(s)\Delta s<\infty$ for some $t_1>0$.

Then Eq.(4.1) has a bounded positive solution,and,for any continuous periodic oscillatory function $\omega(t)$ with period r, has a bounded oscillatory solution

$$x(t)=c^{\frac{t}{r}}(\omega(t)+R(t)) \qquad (4.3)$$

where $|R(t)|<\alpha M$, $\alpha\in(0,1)$.

The proof of Theorem 4.1 is based on the following lemma.

Lemma 4.1. Suppose that the conditions of Theorem 4.1 hold. Then the equations

$$(x(t)-cx(t-r))^{\Delta}+(P(t)(x(t-\theta)+(2\overline{M}+\omega(t-\theta))c^{\frac{t-\theta}{r}})$$
$$-Q(t)(x(t-\delta)+(2\overline{M}+\omega(t-\delta))c^{\frac{t-\delta}{r}})=0 \qquad (4.4)$$

and

$$(x(t)-cx(t-r))^{\Delta}+(P(t)(x(t-\theta)+2\overline{M})c^{\frac{t-\theta}{r}})-Q(t)(x(t-\delta)+2\overline{M})c^{\frac{t-\delta}{r}})=0 \qquad (4.5)$$

have bounded positive solutions $u_1(t)$ and $u(t)$ respectively,such that

$$|u(t)|\le\frac{\alpha}{2}Mc^{\frac{t}{r}},|u_1(t)|\le\frac{\alpha}{2}Mc^{\frac{t}{r}}.$$

Proof. We give only the outline of the proof for Eq.(4.5). We consider the integral equation of the form

$$x(t)=cx(t-r)+\int_{t-\theta}^{t-\delta}Q(s+\delta)(x(s)+2\overline{M}c^{\frac{s}{r}})\Delta s+\int_{t}^{\infty}\overline{P}(s)(x(s-\theta)+2\overline{M}c^{\frac{s-\theta}{r}}\Delta s \qquad (4.6)$$

Letting $z(t)=x(t)c^{-\frac{t}{r}}$,then (4.6) becomes

$$z(t)=z(t-r)+\int_{t-\theta}^{t-\delta}Q(s+\delta)(z(s)+2\overline{M}c^{\frac{s-t}{r}})\Delta s+\int_{t}^{\infty}\overline{P}(s)(x(s-\theta)+2\overline{M}c^{\frac{s-t-\theta}{r}}\Delta s \qquad (4.7)$$

To prove the lemma ,it is sufficient to prove that (4.7) has a bounded positive solution $z(t)$ such that $|z(t)|<\frac{\alpha}{2}M$,for $t\ge t_1$,where t_1 is a sufficiently large number.

Choose t_1 sufficiently large such that

$$\sum_{j=0}^{\infty}\int_{t_1+jr}^{\infty}c^{\frac{s-t_1-jr}{r}}\overline{P}(s)\Delta s+n\int_{t_1}^{\infty}Q(s)\Delta s<\frac{\alpha M}{16\overline{ME}},$$

where $E=c^{-\frac{\theta}{r}}>1$.

The rest of the proof is quite similar to the proof of Lemma 2.2 and we omit it.

In view of Lemma 4.2,we can prove Theorem 4.1 by the similar method to the proof of Theorem 2.1. We omit the detail here.

In the following we give an explicit condition to guarantee that (H8) holds.

Corollary 4.1. If (H1),(H2),(H3),(H5) and the following condition hold.

(H9) $\int^{\infty} \overline{P}(s)\Delta s < \infty,$

Then the conclusion of Theorem 4.1 is true.

Proof. It is sufficient to prove that (H9) implies (H8).

Set $j = \left[\dfrac{t-t_1}{r}\right]$, where $[\cdot]$ denotes the greatest integer function. Then $t - r \le t_1 + jr \le t$ and $t_1 + jr \le t \le t_1 + (j+1)r$.

Let $I = \sum_{j=0}^{\infty}\int_{t_1+jr}^{\infty} c^{\frac{s-t_1-jr}{r}}\overline{P}(s)\Delta s$,then

$$I \le \frac{1}{r}\sum_{j=0}^{\infty}\int_{t_1+jr}^{t_1+(j+1)r}\Delta t\int_{t_1+jr}^{\infty} c^{\frac{s-t_1-jr}{r}}\overline{P}(s)\Delta s$$

$$\le \frac{1}{cr}\sum_{j=0}^{\infty}\int_{t_1+jr}^{t_1+(j+1)r}\Delta t\int_{t-r}^{\infty} c^{\frac{s-t+r}{r}}\overline{P}(s)\Delta s$$

$$= \frac{1}{cr}\int_{t_1}^{\infty}\Delta t\int_{t-r}^{\infty} c^{\frac{s-t+r}{r}}\overline{P}(s)\Delta s$$

$$= \frac{1}{cr}\int_{t_1-r}^{\infty}\Delta t\int_{t}^{\infty} c^{\frac{s-t}{r}}\overline{P}(s)\Delta s$$

$$= \frac{1}{cr}\int_{t_1-r}^{\infty}\overline{P}(s)\Delta s\int_{t}^{\infty} c^{\frac{s-t}{r}}\Delta t$$

$$\le \frac{1}{cr}\left(\int_{t_1-r}^{\infty}\overline{P}(s)\Delta s\right)\left(\int_{t}^{\infty} c^{\frac{u}{r}}\Delta u\right)$$

$$= K\int_{t_1-r}^{\infty}\overline{P}(s)\Delta s,$$

where $K = \dfrac{1}{cr}\int_0^{\infty} c^{\frac{u}{r}}\Delta u$.

Therefore (H9) implies (H8). The proof is complete.

5 Bounded solutions of Eq.(1.2)

Consider Eq.(1.2) with $c = 1$.

$$(x(t) - x(t-r))^{\Delta} = P(t)x(t-\theta) - Q(t)x(t-\delta) \tag{5.1}$$

Theorem 5.1. Assume that (H1) hold. Then Eq.(5.1) has a bounded positive solution and for any periodic oscillatory sequence $\{\omega(t)\}$with period r ,has a bounded oscillatory solution $\{x(t)\}$ such that

$$x(t) = \omega(t) + R(t) \tag{5.2}$$

where $|R(t)| < \alpha M$, $\alpha \in (0,1)$.

The proof of Theorem 5.1 is based on the following Lemma.

Lemma 5.1. Suppose the conditions of Theorem 5.1 hold.THen the equations

$$(x(t) - x(t-r))^{\Delta} = p(t)(x(t-\theta) + 2\overline{M} + \omega(t-\theta)) - q(t)(x(t-\delta) + 2\overline{M} + \omega(t-\delta)) \tag{5.3}$$

And

$$(x(t)-x(t-r))^{\Delta}=p(t)(x(t-\theta)+2\overline{M})-q(t)(x(t-\delta)+2\overline{M})$$

have bounded positive solutions $\overline{u}(t)$ and $u(t)$, respectively ,such that

$$|u(t)|\leq\frac{\alpha}{2}M,|\overline{u}(t)|\leq\frac{\alpha}{2}M,$$

where $\alpha\in(0,1)$.

Proof. THe proof about Eq.(5.3) is quite similar to that about Eq.(5.4).So we give only the proof about Eq.(5.4). Choose t_1 sufficiently large such that (2.5) holds. Define a set Ω by

$$\Omega=\left\{x(t)\in X\middle|0\leq x(t)\leq\frac{\alpha}{4}M,n\geq t_1\right\}$$

and a series of sequences of $\{x^{(l)}(t)\},l=0,1,2,\cdots,$ by $x^{(0)}(t)=0,t\geq t_1$,

$$x^{(l)}(t)=\begin{cases}x^{(l-1)}(t+r)+\int_{t-\theta+\delta+r}^{t+r-1}q(s)(x^{(l-1)}(s-\delta)+2\overline{M})\Delta s\\ \qquad+\int_{t+r}^{\infty}\overline{p}(s)(x^{(l-1)}(s-\theta)+2\overline{M})\Delta s & t\geq t_1+D\\ \qquad x^{(l)}(t+D) & t_1\leq t\leq t_1+D.\end{cases}$$

where $D=\max\{0,\theta-r\}$, $l=1,2,\cdots$.

Clearly, $x^{(l)}(t)>0=x^{(0)}(t),t\geq t_1$.By induction, we have

$$x^{(0)}(t)<\cdots<x^{(l-1)}(t)<x^{(l)}(t)<\cdots,t\geq t_1,l=1,2,\cdots.$$

It is obvious that $x^{(0)}(t)\leq\frac{\alpha}{4}M,t\geq t_1$. Suppose that

$$x^{(l)}(t)\leq\frac{\alpha}{4}M,t\geq t_1,l=1,2,\cdots,s-1.$$

We are going to prove that

$$x^{(s)}(t)\leq\frac{\alpha}{4}M,t\geq t_1,$$

In fact, when $t\leq t_1+D$,

$$\begin{aligned}x^{(s)}(t)&=x^{(s-1)}(t+r)+\int_{t-\theta+\delta+r}^{t+r-1}q(s)(x^{(s-1)}(s-\delta)+2\overline{M})\Delta s+\int_{t+r}^{\infty}\overline{p}(s)(x^{s-1)}(s-\theta)+2\overline{M})\Delta s\\ &=x^{(0)}(t+sr)+\sum_{j=1}^{s}\int_{t-\theta+\delta+jr}^{t+jr-1}q(s)(x^{(s-1)}(s-\delta)+2\overline{M})\Delta s+\sum_{j=1}^{s}\int_{t+jr}^{\infty}\overline{p}(s)(x^{s-1)}(s-\theta)+2\overline{M})\Delta s\\ &\leq\frac{M}{4}(\sum_{j=1}^{s}\int_{t-\theta+\delta+jr}^{t+jr-1}q(s)\Delta s+\sum_{j=1}^{s}\int_{t+jr}^{\infty}\overline{p}(s)\Delta s\\ &\leq\frac{\alpha}{4}M.\end{aligned}$$

Therefore, $\{x^{(l)}(t)\}\subset\Omega,l=1,2,\cdots,$ In view of (5.5), there exists $\{u(t)\}\subset\Omega$ such that $\lim_{l\to\infty}x^{(l)}(t)=u(t),t\geq t_1$. So

$$u(t)=\begin{cases}u(t+r)+\int_{t-\theta+\delta+r}^{t+r-1}q(s)(u(s-\delta)+2\overline{M})\Delta s\\ \qquad+\int_{t+r}^{\infty}\overline{p}(s)(x^{(l-1)}(s-\theta)+2\overline{M})\Delta s & t\geq t_1+D\\ \qquad u(t+D) & t_1\leq t\leq t_1+D.\end{cases}$$

i.e., $\{u(t)\}$ is a solution of Eq.(5.4).The proof is complete.
In view of Lemma 5.1,we can prove Theorem 5.1 by the similar method to the proof of Theorem 2.1.We omit it. Similar to Theorem 2.2, we have

Theorem 5.2. Assume that (H2) holds. If Eq.(5.1) has a bounded positive solution $\{x(t)\}$ such that $\liminf_{n\to\infty} x(t) > 0$, then (H1) holds.

Similar to Theorem 3.1 and Theorem 3.3, we have the following results.

Theorem 5.3. Assume that (H3),(H4) hold. Then Eq.(5.1) has an A-type positive solution .

Theorem 5.4. Assume that (H1),(H4) and (H5) hold. Then Eq.(5.1) has a B-type positive solution .

Now we consider Eq.(1.2)with $c \in (0,1)$ .Similar to Corollary 4.1,we have

Theorem 5.5. Suppose that $c \in (0,1)$,(H2) and (H6) hold. Then Eq.(1.2) has a bounded positive solution, and for any periodic oscillatory sequence $\{\omega(t)\}$ with period m, has a bounded oscillatory solution

$$x(t) = c^{\frac{n}{m}}(\omega(t) + R(t)) ,$$

where $|R(t)| < \alpha M$, $\alpha \in (0,1)$.

6 Unbounded solutions of Eq.(1.1) and Eq(1.2) with $c > 1$

Similarly,in the case $c > 1$,we have the following result.

Theorem 6.1. Suppose that $c > 1$. (H1),(H2),(H3),(H5) and (H8) hold. Then Eq.(1.1) and Eq.(1.2) has an unbounded positive solution, and,for any continuous periodic oscillatory function $\omega(t)$ with period r, has an unbounded oscillatory solution

$$x(t) = c^{\frac{t}{r}}(\omega(t) + R(t)) ,$$

where $|R(t)| < \alpha M$, $\alpha \in (0,1)$.

In the following we give an explicit condition to guarantee that (H8) holds in the case $c > 1$.

Corollary 6.1. If $c > 1$,(H1),(H2),(H3),(H5) and the following condition hold.

(H10) $\int^{\infty} c^{\frac{s}{t}} \overline{P}(s)\Delta s < \infty,$

Then the conclusion of Theorem 6.1 is true.

Proof. It is sufficient to prove that (H10) implies (H8).

Set $j = \left[\frac{t-t_1}{r}\right]$,then $t - r \le t_1 + jr \le t$ and $t_1 + jr \le t \le t_1 + (j+1)r$.

Let $I = \sum_{j=0}^{\infty}\int_{t_1+jr}^{\infty} c^{\frac{s-t_1-jr}{r}} \overline{P}(s)\Delta s$, then

$$\begin{aligned} I &\le \frac{1}{r}\sum_{j=0}^{\infty}\int_{t_1+jr}^{t_1+(j+1)r}\Delta t\int_{t_1+jr}^{\infty} c^{\frac{s-t_1-jr}{r}} \overline{P}(s)\Delta s \\ &= \frac{1}{r}\int_{t_1}^{\infty}\Delta t\int_{t-r}^{\infty} c^{\frac{s-t+r}{r}} \overline{P}(s)\Delta s \\ &= \frac{1}{r}\int_{t_1-r}^{\infty}\Delta t\int_{t}^{\infty} c^{\frac{s-t}{r}} \overline{P}(s)\Delta s \\ &= \frac{1}{r}\int_{t_1-r}^{\infty}\overline{P}(s)\Delta s\int_{t_1-r}^{s} c^{\frac{s-t}{r}}\Delta t \end{aligned}$$

$$= \frac{1}{r}\int_{t_1-r}^{\infty} \overline{P}(s)\Delta s \int_0^{s-t_1+r} c^{\frac{u}{r}}\Delta u$$

$$\leq \frac{1}{r}\int_{t_1-r}^{\infty} \overline{P}(s)\Delta s((\ln c)^{-1} r c^{\frac{s-t_1-r}{r}}\Big|_0^{s-t_1-r})$$

$$\leq \frac{1}{\ln c}\int_{t_1-r}^{\infty} c^{\frac{s-t_1-r}{r}} \overline{P}(s)\Delta s$$

$$= K\int_{t_1-r}^{\infty} c^{\frac{s}{r}} \overline{P}(s)\Delta s \,,$$

where $K = (\ln c \cdot c^{\frac{t_1+r}{r}})^{-1}$.

Therefore (H10) implies (H8). The proof is complete.

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