# Laplace Transformation Techniques vs. Extended Semi-Markov Processes Method

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## Abstract

*We consider a continuously monitored one-unit system supported by an identical spare unit, and perfectly repaired by an in-house regular repairer within a predetermined fixed patience time, or by a visiting expert repairer who arrives when the system fails or when the patience time is over, and who repairs all failed units. We demonstrate the difficulties and the shortcomings of the Laplace transformation technique, and how these are overcome by the extended semi-Markov process method.*

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**Keywords:** Repairable system; Laplace transformation; Extended semi-Markov process; Limiting availability

## *1. Introduction*

In the literature, most repairable models are studied by using the Laplace transformation technique. (See, for example, KumarGupta and Taneja (1995), Osaki and Asakura (1970), Sen and Bhattacharjee (1986), Sarkar and Chaudhuri (1999), Sridharan and Mohanavadivu (1998), Sridharan (2000), Wang, Ke and Lee (2007), Zhang and Wang (2007).) The majority of these models assume exponential life-times or/and exponential repair-times. Typically, one establishes a system of renewal-type equations which rely on the memoryless property from exponential distribution. Then the Laplace transformation technique is used to solve the system. Nonetheless, the same task can be accomplished more efficiently by using the method of semi-Markov process (SMP). But neither method can accommodate the more realistic arbitrary lifetime distribution. Recently, Bieth, Hong and Sarkar (See Bieth, Hong and Sarkar (2009), Bieth, Hong and Sarkar (2010)) introduced an extended semi-Markov process (ESMP) method to solve stochastic repairable models that allow arbitrary life- and repair- times. They extend the limiting probability theorem of an SMP to that of an ESMP. The approach is applicable to a wide spectrum of situations.

In this paper, we attempt to study the repairable model in Bieth, Hong and Sarkar (2010) under arbitrary life and repairable times, by employing the traditional Laplace transformation technique. We demonstrate only a partial success: The Laplace transformation technique derives a formal solution to the limiting availability, though the mathematical manipulation is quite complicated. However, it seems to be a formidable challenge to derive the limiting proportion of times the system spends in various states, which is essential for carrying out a cost analysis. On the other hand, this model can be solved completely using the ESMP method, including determination of the limiting proportion times in each state. Thus, the EMSP method not only yields the limiting availability more conveniently, but also it offers the option to conduct cost analysis. The remaining of this paper are organized as follows. Section 2 introduces the model. Section 3 establishes the renewal-type equations. Section 4 provides a formal solution to the limiting availability using the Laplace transformation technique. Section 5 specializes the formal solution to the case of the exponential life- and repair- times, as an example. Section 6 summarizes the solution using the ESMP method. Section 7 concludes the paper with a brief discussion.

## 2. Model Setting

We consider a continuously monitored two-identical-unit cold standby system. As soon as the operating unit fails it undergoes repair by an in-house regular repairman, while the spare is placed on operation immediately. The regular repairer is allowed a predetermined fixed patience time $T$ to complete repair. If the operating unit fails while the other unit is still under repair, the system fails. A visiting expert repair person is called in as soon as either the patience time is over or the system fails. The repair facility can accommodate only one repairman at a time, and the benefit of partial repair done by the regular repairer is forfeited when the expert takes over. For each trip, the expert repairs all failed unit before he leaves. A freshly failed unit awaits repair while the previously failed unit is being repaired. The two repairmen take stochastically different amounts of time to complete repair, after which the unit becomes as good as new. Variations of this model have been studied by several authors. (See, for example, Kumar Gupta and Taneja (1995), Sridharan and Mohanavadivu (1998), and Tuteja, Arora, and Taneja (1991).) These papers assume the lifetime is exponentially distributed. They rely on the memoryless property of exponential distribution and use the Laplace transformation technique to obtain time-dependent availability, reliability, busy periods for the two repairers and total profit. On the other hand, Bieth, Hong and Sarkar (2009) shows that for exponential life- and exponential repair time distributions, the steady-state probabilities can be obtained by recognizing that the continuous time stochastic process (CTSP) is in fact a semi-Markov process (SMP). Neither method can accommodate arbitrary life- and repair- times. That more realistic problem is solved in Bieth, Hong and Sarkar (2010), which systematically develops the ESMP method.

Consistent with the notation of Bieth, Hong and Sarkar (2009) and Bieth, Hong and Sarkar (2010), we let $T$ denote the patience time and let the lifetime $X$ of an operating unit have a cumulative distribution function (CDF) $F$, the repair time $Y_r$ by the regular repair person have a CDF $G_r$, and the repair time $Y_e$ by the expert repair person have a CDF $G_e$. Let the corresponding survival functions (sf) be $\overline{F}$, $\overline{G}_r$, $\overline{G}_e$. Let each of these random variables be absolutely continuous with probability density functions (pdf) $f$, $g_r$, $g_e$ respectively. We assume all lifetimes and repair times are stochastically independent. At any instant the status of a unit is $s$ (on standby), $p$ (in operation), $r$ (under repair by the regular repair person), $e$ (under repair by the expert), or $w$ (awaiting repair). Consequently, depending on the status of the two units, the system is in one of the following four states: $0=(s,p)$, $1=(r,p)$, $2=(e,p)$, and $3=(e,w)$. The system is up in states 0, 1, and 2 and down in state 3. Since the two units are identical in their stochastic behavior, it is irrelevant which unit is on operation and which is under repair. Also, when both units are down, they are repaired in the order in which they failed.

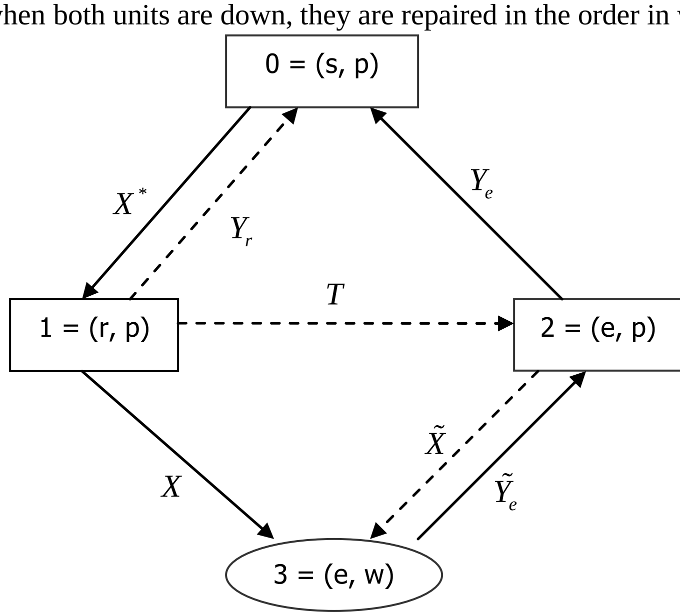


Figure 1: The schematic diagram of transitions

Figure 1 gives the schematic diagram of transitions from one state to another. At time $t=0$ the system starts in state 0. It stays there for a random duration $X^* = X$ , and then it enters state 1. The sojourn time in state 1 is the smallest among $Y_r, X, T$ .The system moves to state 0 if $Y_r$ is the smallest, to state 3 if $X$ is the smallest, or to state 2 if $T$ is the smallest. The sojourn time in state 2 is the smaller of $Y_e$ and $\tilde{X}$ , where $\tilde{X}$ equals $X-T$ if the system had arrived at state 2 from state 1 and it equals $X$ if from state 3. From state 2 the system moves to state 0 if $Y_e < \tilde{X}$ , or to state 3 otherwise. From state 3 the system can only move to state 2 as soon as the expert finishes repair on the unit that has failed earlier. Hence, the sojourn time in state 3 is $\tilde{Y}_e$ , which equals $Y_e$ if the system had arrived at state 3 from state 1 and it equals $Y_e - \tilde{X}$ if from state 2. Finally, the sojourn time in state 0 (except for the initial start at time $t=0$) is $X^* = X - Y_r$ if the system had arrived at state 0 from state 1, and it is $X^* = \tilde{X} - Y_e$ if from state~2. From state 0 the system always moves to state 1. (In this paragraph, wherever two random variables are equated, we mean that they have the same CDF.)

### *3. Renewal-type Equations*

A full description of the continuous time stochastic process requires documenting exactly how long each unit is in operation/under repair/awaiting repair. However, we shall concentrate on an embedded discrete-time stochastic process (DTSP) by focusing attention to the epochs when one unit is just put on repair (by the regular or the expert repair person) and the other unit just starts to operate or to wait for repair. We shall not keep track of other epochs when transition from one state to another takes place. To be precise, our DTSP keeps track of all epochs when the system enters state $1=(r,p)$ (albeit from state $0=(s,p)$). But it completely bypasses all epochs when the system enters state 0 either from state 1 or from state $2=(e,p)$, as that necessitates keeping record of the ongoing operating time. The DTSP keeps tracks of epochs when the system goes down by entering state $3=(e,w)$ from state 1, but it ignores all epochs when the system goes down by entering state 3 from state 2, since that requires keeping record of the ongoing repair time. Finally, the DTSP keeps track of epochs when the down system is revived and it enters state 2 from state 3, but it ignores all epochs when the system enters state 2 from state 1.

By focusing attention on the above-mentioned DTSP we are able to construct a system of renewal-type equations. For $(\delta_1,\delta_2)=(s,p),(r,p),(e,p),(e,w)$, let $B_{\delta_1,\delta_2}(t)$ denote the probability that the system is down $t$ units of time after the epoch when it just enters state $(\delta_1,\delta_2)$ . Then clearly, $A(t)=1-B_{s,p}(t)$ .In order to find expression for $B_{s,p}(t)$ , we state, justify and solve the following system of four integral equations involving $B_{s,p}(t)$ , $B_{r,p}(t)$ , $B_{e,p}(t)$ and $B_{e,w}(t)$ .

$$B_{s,p}(t)=\int_0^t B_{r,p}(t-x)dF(x) \tag{3.1}$$

$$B_{r,p}(t)=\begin{cases} \int_0^t B_{r,p}(t-x)G_r(x)dF(x)+\int_0^t B_{e,w}(t-x)\overline{G}_r(x)dF(x), & if\ t\le T,\\ \int_0^T B_{r,p}(t-x)G_r(x)dF(x)+\int_0^T B_{e,w}(t-x)\overline{G}_r(x)dF(x), & \\ \quad +\int_T^t B_{r,p}(t-x)\left[G_r(T)+\overline{G}_r(T)G_e(x-T)\right]dF(x) & \\ \quad +\overline{G}_r(T)\int_T^t B_{e,p}(t-y)\left[F(y)-F(T)\right]dG_e(y-T), & \\ \quad +\overline{G}_r(T)\overline{G}_e(t-T)\left[F(t)-F(T)\right], & if\ t>T \\ . & \end{cases} \tag{3.2}$$

$$B_{e,p}(t)=\int_0^t B_{r,p}(t-x)G_e(x)dF(x)+\int_0^t B_{e,p}(t-y)F(y)dG_e(y)+F(t)\overline{G}_e(t) \tag{3.3}$$

$$B_{e,w}(t)=\int_0^t B_{e,p}(t-y)dG_e(y)+\overline{G}_e(t) \tag{3.4}$$

The justification for integral equations (3.1)-(3.4) follows along similar lines. For example, we justify (3.2) for $t > T$ as follows: If the operating unit fails at time $x \in (0, T]$, depending on whether or not the regular repair person finishes repair by time *x*, the system just enters state 1 or 3. This accounts for the first two integrals on the right hand side (rhs). On the other hand, if the failure happens at time $x \in (T, t]$ and the repair is complete during $(0, x)$, which happens with probability $G_r(T) + \overline{G}_r(T) G_e(x - T)$ since at time $T$ the expert repair person replaces the regular one, then the system just enters state 1 at time *x*.

This contributes the third integral on the rhs. The fourth integral on the rhs comes from the event that the failure happens at time $x \in (T, t]$ and the repair is completed afterwards at time $y \in (x, t)$. Surely, this event implies that the regular repair person did not complete repair by time *T*, which has probability $\overline{G}_r(T)$, and the expert repair person finished repair in $y - T$ additional time. The event also implies that the failure occurred during $(T, y]$, which happens with probability $F(y) - F(T)$; and the system just enters state 2 at time *y*. Finally, the last term on the rhs accounts for the event that failure did happen sometime in $(T, t)$ with probability $F(t) - F(T)$, but the repair was not completed by the regular repair person by time *T*, nor by the expert in $t - T$ additional time.
Note we may rewrite the integral equation (3.2) in the following form:

$$\begin{aligned} B_{r,p}(t) &= \int_0^t B_{r,p}(t-x) G_1(x) dF(x) + \int_0^t B_{e,w}(t-x) \overline{G}_0(x) dF(x) \\ &+ \overline{G}_r(T) \int_0^t B_{e,p}(t-y) \left[F(y) - F(T)\right] dG_e(y-T), \qquad (3.5) \\ &+ \overline{G}_r(T) \overline{G}_e(t-T) \left[F(t) - F(T)\right]^+, \end{aligned}$$

where

$$G_1(x) = \begin{cases} G_r(x) & \text{if } x \le T \\ G_r(T) + \overline{G}_r(T) G_e(x-T) & \text{if } x > T \end{cases}$$

$$G_0(x) = \begin{cases} G_r(x) & \text{if } x \le T \\ 1 & \text{if } x > T \end{cases}$$

This can be justified as follows: When $y \le T$, we have $dG_e(y-T) = 0$ and $[F(t) - F(T)]^+$. So, equation (3.5) reduces to (3.2). When $t > T$, the first integral on the rhs of (3.5) can be written as $\int_0^T B_{r,p}(t-x) G_1(x) dF(X) + \int_T^t B_{r,p}(t-x) G_1(x) dF(x)$, which by definition of $G_1$ equals $\int_0^T B_{r,p}(t-x) G_r(x) dF(X) + \int_T^t B_{r,p}(t-x) \left[G_r(T) + \overline{G}_r(T) G_e(x-T)\right] dF(x)$. Also note that $\overline{G}_0(x) = 0$ for $x \ge T$, so that the second integral on the rhs of (3.5) equals $\int_0^T B_{e,w}(t-x) \overline{G}_r(x) dF(x)$. Thus, the first two integrals on the rhs of (3.5) together equal the first three integrals on the rhs on (3.2). The remaining terms on the rhs of (3.5) obviously equal the remaining terms on the rhs of in (3.2).

## 4. Solution to $A_\infty$

We use the fact that the Laplace transformation of a convolution of two functions equals the product of their Laplace transformations. Taking Laplace transformations in equations (3.1), (3.2), (3.3), (3.4), we obtain (by suppressing the argument *s*)

$$B^*_{s,p} = B^*_{r,p} \; f^* \qquad (4.6)$$

$$B^*_{r,p} = B^*_{r,p} \, [fG_1]^* + B^*_{e,w} [f\overline{G}_0]^* + \overline{G}_r(T) B^*_{e,p} [g_e^{>} F^{\vee}]^* + \overline{G}_r(T) [\overline{G}_e^{>} (F^{\vee})^+]^* \qquad (4.7)$$

$$B^*_{e,p} = B^*_{r,p} \, [fG_e]^* + B^*_{e,p}(t) [g_e F]^* + [F\overline{G}_e]^* \qquad (4.8)$$

$$B^*_{e,w} = B^*_{e,p}(t) g_e^* + (\overline{G}_e)^* \qquad (4.9)$$

where $F^{\vee}$ is a shift of $F(x)$ downward by $F(T)$ and $g_e^{>}$, $G_e^{>}$ are shifts of $g_e(x)$, $G_e(x)$ of the right by *T* respectively; that is,

$$F^{\vee}(x) = F(x) - F(T)$$
$$g_e^{>}(x) = g_e(x - T)$$
$$G_e^{>}(x) = G_e(x - T).$$

We first solve the system consisting of equations (4.7)-(4.9). Putting those three equations in matrix form, we have

$$\begin{pmatrix} B^*_{r,\,p} \\ B^*_{e,\,p} \\ B^*_{e,\,w} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & 0 \\ 0 & a_{32} & 0 \end{pmatrix} \begin{pmatrix} B^*_{r,\,p} \\ B^*_{e,\,p} \\ B^*_{e,\,w} \end{pmatrix} + \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

where

$$a_{11} = [fG_1]^*, \qquad a_{12} = \overline{G}_r(T)[g_e^{>} F^{\vee}]^*, \qquad a_{13} = [f\overline{G}_0]^*,$$
$$a_{21} = [fG_e]^*, \qquad a_{22} = [g_e F]^*, \qquad a_{32} = [g_e]^*,$$
$$c_1 = \overline{G}_r(T)[\overline{G}_e^{>}(F^{\vee})^{+}]^*, \qquad c_2 = [F\overline{G}_e]^*, \qquad c_3 = \left(\overline{G}_e\right)^*, \qquad (4.10)$$

Solving this system for $B^*_{r,\;p}$ and substituting it in (4.6), we obtain

$$B^*_{s,\;p} = \frac{f^*\left[(1-a_{22})c_1 + (a_{12}+a_{13}a_{32})c_2 + a_{13}(1-a_{22})c_3\right]}{\left[(1-a_{11})(1-a_{22}) - a_{12}a_{21} - a_{13}a_{21}a_{32}\right]} \qquad (4.11)$$

Note that $sA^*(s) = s\int_0^\infty e^{-st}A(t)dt = -\int_0^\infty A(t)de^{-st} = A(0) + \int_0^\infty e^{-st}A'(t)dt$. Hence, taking limit as $s$ approaches 0 and interchanging limit and integral, we have

$$\lim_{s\to 0} sA^*(s) = A(\infty) = \lim_{t\to\infty} A(t)$$

Since $A(t) = 1 - B_{s,\,p}(t)$, we have $A^*(s) = \dfrac{1}{s} - B^*_{s,\,p}(s)$. Therefore,

$$\begin{aligned} A(\infty) &= \lim_{s\to 0} sA^*(s) \\ &= \lim_{s\to 0} s\left[\frac{1}{s} - B^*(s)\right] \\ &= 1 - \lim_{s\to 0} sB^*_{s,\,p}(s). \end{aligned}$$

## 5. *Example*

**Example 1** (Exponential lifetime, exponential repair times). Suppose that $X, Y_r, Y_e$ have exponential distributions with scale parameters $\lambda$, $\mu_r$, $\mu_e$ respectively. Without loss of generality, we may assume $\mu_r \neq \mu_e$. In this case equation (4.10) yields

$$a_{11} = \lambda\left[\frac{1}{s+\lambda} - \frac{1}{s+\lambda+\mu_r}\left[1 - e^{-(s+\lambda+\mu_r)T}\right] - \frac{1}{s+\lambda+\mu_e}e^{-(s+\lambda+\mu_r)T}\right]$$

$$a_{12} = \frac{\lambda\mu_e}{(s+\mu_e)(s+\lambda+\mu_e)}e^{-(s+\lambda+\mu_r)T}$$

$$a_{13} = \frac{\lambda}{s+\lambda+\mu_r}\left[1 - e^{-(s+\lambda+\mu_r)T}\right]$$

$$a_{21} = \frac{\lambda\mu_e}{(s+\lambda)(s+\lambda+\mu_e)}$$

$$a_{22} = \frac{\lambda\mu_e}{(s+\mu_e)(s+\lambda+\mu_e)}$$

$$a_{32} = \frac{\mu_e}{s+\mu_e}$$

$$c_1 = \frac{\lambda}{(s+\mu_e)(s+\lambda+\mu_e)} e^{-(s+\lambda+\mu_e)T}$$

$$c_2 = \frac{\lambda}{(s+\mu_e)(s+\lambda+\mu_e)}$$

$$c_3 = \frac{1}{s+\mu_e}.$$

Substituting these in (4.11), and then using (4.12), we get

$$A_\infty = \left[1 + \frac{\lambda}{\mu_e}\left\{1 + \frac{\mu_e}{\lambda}\left(1 + \frac{(\mu_e - \mu_r)(1-\gamma^T)}{\lambda + \mu_r}\right)^{-1}\right\}^{-1}\right]^{-1}. \quad (5.13)$$

where $\gamma = e^{-(\lambda+\mu_r)}$. This agrees with the results in Bieth, Hong and Sarkar (2009).

## *6. The ESMP Method*

We describe the ESMP method in general terms first. Then we apply the ESMP method to the repairable model studied in this paper. Finally, we specialize it to Example 1. Under the assumption of arbitrary life- and repair-times, oftentimes the CTSP is no longer an SMP because the embedded DTSP that tracks every transition is not Markovian. That is, the sojourn time in a state and the transition probabilities out of this state may depend on the previous state(s). However, when one can identify another DTSP which is Markovian (that is, both the sojourn time in a state and the transition probabilities out of this state depend only on the current state), then the CTSP is called an ESMP.

For an ESMP, Theorem 1 below gives a method of computing the probability the CTSP spends in various states. We need some notation. Suppose that we have an ESMP with state space *S* such that the DTSP restricted to only recorded transitions is a positive recurrent Markov chain with state space $S' \subset S$ and transition matrix $\mathbf{P} = (P_{i\,j})$. The transition probabilities are obtained by listing all possible paths $\phi = (i_1, i_2, ..., i_m)$, consisting of a set of states $\{i_1, i_2, ..., i_m\}$ (this set may be empty) the visits to which are not recorded, that the CTSP can follow starting from a recorded visit to state *i* to reach the next recorded visit to state *j* (which may be the same as state *i*). Let $P_{i(\phi)j}$ denote the conditional probability that, starting from a recorded visit to state *i*, the CTSP moves along path $\phi$ to make the next recorded visit to state *j*. Adding over all paths between a recorded visit to state *i* and the next recorded visit to state *j*, we have

$$P_{i\,j} = P_{i(\phi)j} \quad (6.13)$$

Note that $\sum_{j \in S'} P_{i\,j} = 1$ for all $j \in S'$, since $\mathbf{P}$ is a stochastic matrix.

The stationary distribution for this DTSP, restricted to only recorded transitions, is obtained (see, for example, Ross (1996), pp. 175-177) by solving the system of equations

$$\pi_j = \sum_{i \in S'} \pi_i P_{ij} , \quad \sum_{j \in S'} \pi_j = 1 \tag{6.14}$$

Note that $\pi_i$ is the long run probability that the transitions enter (or exit) state *i*.

Let $\mu_{k|i(\phi)j}$ denote the expected time the CTSP spends in state $k = i, i_1, i_2, ..., i_m$ as it moves from a recorded visit to state *i* to the next recorded visit to state *j* via $\phi$. Then we have the following theorem. (See Bieth, Hong and Sarkar (2010) for a detailed proof.)

**Theorem 1** *For an ESMP, if the transition matrix* $P = (P_{ij})_{i,\, j \in S'}$ *of visits that are recorded is positive recurrent with stationary distribution* $\pi = (\pi_i)_{i \in S'}$ *, then the limiting proportion of time the ESMP spends in any state* $k \in S$ *(visit to which may be recorded or unrecorded) is given by*

$$\theta_k \propto \sum_{i \in S'} \left( \sum_{j \in S'} \sum_{\phi} P_{i(\phi)j} \mu_{k|i(\varphi)j} \right) \tag{6.15}$$

The proportionality constant is obtained from the constraint $\sum_{k \in S} \theta_k = 1$ . Thereafter, the limiting availability, which is the long-run proportion of times the system is up, can be obtained as $A_\infty = \sum_{k \in U} \theta_k$ , where $U \subset S$ consists of all the up states.

Under the model studied in this paper, the stochastic behavior of the system is a CTSP with state space {0, 1, 2, 3}, of which state 3 is the only down state. Indeed, this CTSP is an ESMP, as explained in the next paragraph. Hence, applying Theorem 1, we obtain expressions for $\theta_k$ , the proportion of time the CTSP spends in state *k* (*k*=0, 1, 2, 3). Thereafter, we obtain the limiting availability as

$$A_\infty = 1 - \theta_3 = \theta_0 + \theta_1 + \theta_2 = \left[ 1 + \frac{\theta_3}{\theta_0 + \theta_1 + \theta_2} \right]^{-1} . \tag{6.16}$$

Likewise, the limiting proportion of busy time for the regular repairer is $\theta_2$ , and that for the expert repairer is $\theta_2 + \theta_3$ . Other useful quantities can be computed using the $\theta_k$ 's .

How is our CTSP an ESMP? We must exhibit a DTSP that is Markovian. Consider as recorded transitions those epochs when one unit is just put on repair (by the regular or the expert repair person) and the other unit just starts to operate or to wait for repair. To be precise, all epochs when the system enters state 1 (albeit from state 0) are recorded, but all epochs when the system enters state 0 either from state 1 or from state 2 are unrecorded, since the sojourn time in state 0 depends on the age of the operating unit (which equals the time spent in the previous state). Also epochs when the system enters state 3 from state 1 are recorded, but epochs when the system enters state 3 from state 2 are not, since the sojourn time in state 3 is the additional time the expert needs to finish the repair she has started in state 2. Finally, epochs when the down system is revived and it enters state 2 from state 3 are recorded, but epochs when the system enters state 2 from state 1 are not recorded. Then the DTSP restricted to recorded transactions only form a Markov process on $S' = \{1, 2, 3\}$. That is, both the sojourn time and the transition probabilities between two successive recorded transitions depend only on the current state. Indeed, this is exactly the same DTSP that we utilized to develop the renewal-type equations of Section 3.

Next, we apply Theorem 1 to our ESMP. The transition paths, the conditional probabilities and the conditional expected sojourn times the CTSP spends in each state along these paths are given in Table 1, where $Z = \min\{X, Y_r, T\}$ and

$$B_r^1 = \{Z = Y_r\}, \qquad B_{Te}^1 = \{Z = T, Y_e < X - T\},$$

$$B_{Tx}^{1} = \{Z = T, Y_e > X - T\}, \qquad B_x^1 = \{Z = X\},$$

$$B_e^2 = \{Y_e < X\}, \qquad B_x^2 = \{Y_e > X\}.$$

**Table 1: Arbitrary continuous life- and repair times**

| Path $i(\phi)j$ | $P_{i(\phi)j}$ | Conditional expected time in state 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|
| 1(0)1 | $P(B_r^1)$ | $E[X - Y_r \mid B_r^1]$ | $E[Y_r \mid B_r^1]$ | 0 | 0 |
| 1(2,0)1 | $P(B_{Te}^1)$ | $E[X - T - Y_e \mid B_{Te}^1]$ | $T$ | $E[Y_e \mid B_{Te}^1]$ | 0 |
| 1(2,3)1 | $P(B_{Tx}^1)$ | 0 | $T$ | $E[X - T \mid B_{Tx}^1]$ | $E[Y_e - X + T \mid B_{Tx}^1]$ |
| 13 | $P(B_x^1)$ | 0 | $E[X \mid B_x^1]$ | 0 | 0 |
| 2(0)1 | $P(B_e^2)$ | $E[X - Y_e \mid B_e^2]$ | 0 | $E[Y_e \mid B_e^2]$ | 0 |
| 2(3)2 | $P(B_x^2)$ | 0 | 0 | $E[X \mid B_x^2]$ | $E[Y_e - X \mid B_x^2]$ |
| 32 | 1 | 0 | 0 | 0 | $E[Y_e]$ |

Next, from Table 1, we get the transition matrix for the Markovian DTSP as

$$P = \begin{pmatrix} P\{B_r^1\} + P\{B_{Te}^1\} & P\{B_{Tx}^1\} & P\{B_x^1\} \\ P\{B_e^2\} & P\{B_x^2\} & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Solving (6.14), the stationary distribution for the Markovian DTSP is

$$\begin{aligned} \pi_1 &= D^{-1} P_{21} \\ \pi_2 &= D^{-1}(1 - P_{11}) \\ \pi_3 &= D^{-1} P_{21} P_{13} \end{aligned}$$

where $D = 1 - P_{11} + P_{21}(1 + P_{13})$.

Thereafter, using Theorem 1 and the fact that $P\{B\}E[W \mid B] = E[WI(B)]$, (where *I(B)* is the indicator function of event *B* which equals 1 on *B* and 0 on $B^c$), we have

$$\theta_0 \propto \pi_1 \left( E[(X - Y_r) I(B_r^1)] + E[(X - T - Y_e) I(B_{Te}^1)] \right) + \pi_2 E[(X - Y_e) I(B_e^2)]$$
$$\theta_1 \propto \pi_1 \left( E[Y_r I(B_r^1)] + E[X I(B_x^1)] + TP\{Z = T\} \right) = \pi_1 E[Z]$$
$$\theta_1 \propto \pi_1 \left( E[Y_e I(B_{Te}^1)] + E[(X - T) I(B_{Tx}^1)] \right) + \pi_2 E[\min\{X, Y_e\}]$$
$$\theta_3 \propto \pi_1 E[(Y_e - X + T) I(B_{Tx}^1)] + \pi_2 E[(Y_e - X) I(B_x^2)] + \pi_3 E[Y_e]$$

where the proportionality constant is the sum of the expressions on the right hand side.

**Example 1 Revisited**. Suppose that $X, Y_r, Y_e$ have exponential distributions with scale parameters $\lambda, \mu_r, \mu_e$ respectively. Then Table 1 specializes to the following Table 2.

**Table 2: Exponential life- and exponential repair times**

| Path $i(\phi)j$ | $P_{i(\phi)j}$ | Conditional expected time in state 0 | 1 | 2 | 3 |
|---|---|---|---|---|---|
| 1(0)1 | $\frac{\mu_r(1-\gamma^T)}{\lambda+\mu_r}$ | $\frac{1}{\lambda}$ | $\frac{1}{\lambda+\mu_r}-\frac{T\gamma^T}{1-\gamma^T}$ | 0 | 0 |
| 1(2,0)1 | $\frac{\mu_e\gamma^T}{\lambda+\mu_e}$ | $\frac{1}{\lambda}$ | $T$ | $\frac{1}{\lambda+\mu_e}$ | 0 |
| 1(2,3)1 | $\frac{\lambda\gamma^T}{\lambda+\mu_e}$ | 0 | $T$ | $\frac{1}{\lambda+\mu_e}$ | $\frac{1}{\mu_e}$ |
| 13 | $\frac{\lambda(1-\gamma^T)}{\lambda+\mu_r}$ | 0 | $\frac{1}{\lambda+\mu_r}-\frac{T\gamma^T}{1-\gamma^T}$ | 0 | 0 |
| 2(0)1 | $\frac{\mu_e}{\lambda+\mu_e}$ | $\frac{1}{\lambda}$ | 0 | $\frac{1}{\lambda+\mu_e}$ | 0 |
| 2(3)2 | $\frac{\lambda}{\lambda+\mu_e}$ | 0 | 0 | $\frac{1}{\lambda+\mu_e}$ | $\frac{1}{\mu_e}$ |
| 32 | 1 | 0 | 0 | 0 | $\frac{1}{\mu_e}$ |

Next, using (6.17) it is easy to verify that

$$\theta_0 = D_\theta^{-1}\frac{1}{\lambda}, \qquad \theta_1 = D_\theta^{-1}\frac{1-\gamma^T}{\lambda+\mu_r},$$

$$\theta_2 = D_\theta^{-1}\frac{1}{\mu_e}\frac{\lambda+\mu_r\gamma^T}{\lambda+\mu_r}, \qquad \theta_3 = D_\theta^{-1}\frac{\lambda}{\mu_e^2}\frac{\lambda+\mu_r\gamma^T+\mu_e(1-\gamma^T)}{\lambda+\mu_r},$$

where

$$D_\theta = \frac{1}{\lambda}+\frac{1}{\mu_e}\left(1+\frac{\lambda}{\mu_e}\right)\frac{\lambda+\mu_r\gamma^T+\mu_e(1-\gamma^T)}{\lambda+\mu_r}.$$

Lastly, using equation (6.16), we have

$$A_\infty = \left[1+\frac{\lambda}{\mu_e}\left\{1+\frac{\mu_e}{\lambda}\left(1+\frac{(\mu_e-\mu_r)(1-\gamma^T)}{\lambda+\mu_r}\right)^{-1}\right\}^{-1}\right]^{-1},$$

consistent with equation (5.13) obtained using the Laplace transformation technique.

## *7. Discussion*

In this paper, we study a stochastic repairable model using the traditional Laplace transformation technique. Using a Markovian DTSP, we are able to set up a system of renewal-type equations (3.1)-(3.4), and obtain a formal solution of the limiting availability. By specializing to the exponential life- and repair times, we reproduce the limiting availability derived in Bieth, Hong and Sarkar (2009). However, to derive the limiting proportion of times spent in each state seems to be a formidable challenge using this traditional approach. In general, setting up such renewal-type equations for other more complicated stochastic models is likely to be rather challenging and solving the system promises to be quite formidable.

On the other hand, having obtained the Markovian DTSP, it is easy to apply the ESMP method, introduced in Bieth, Hong and Sarkar (2009) and Bieth, Hong and Sarkar (2010). Aside from computational convenience, the ESMP method yields the limiting proportion of times the CTSP spends in each state. Therefore, we can not only obtain the limiting availability, but also obtain the necessary ingredients for carrying out a cost-benefit analysis. Thus, in general, the ESMP method has several advantages over the Laplace transformation technique.

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