

An Easy Algorithm for Students to Generate Implicit N-Holed Tori in Maple and Mathematica

Joshua J. Leiter¹²

Abstract

The purpose of this paper is to provide a pedagogical approach on how to make an n -holed torus in Maple and Mathematica for undergraduate students. It shows how to 3D-print out an N -holed torus from only a mathematical equation. This eliminates the hassle of fudging with parameters or drilling the holes yourself. An algorithm to generate N -Holed tori is provided with all parameters for reasonable bounds and parameters given. A specific application of this algorithm in Mathematica shows the print out of a seven-holed torus in figure 9. Furthermore, this method automatically adds in a discrete symmetry determining where the tori's holes occur. Human beings like symmetry and the discrete symmetry in the algorithm makes the surface aesthetically pleasing. Hopefully the examples generated from this method will allow for: fun 3D printing, and provide simple examples useful for teaching 2D implicit geometry and topology. An explicit construction of a 64-holed torus is given.

1 Introduction, what is a torus?

There are several things already known about tori and how to plot them. Implicit surfaces in Maple and Mathematica have numerous tools to support their generation such as “implicitplot3d” in Maple, and “ContourPlot3D” in Mathematica. A reference for implicit surfaces can be found in [1]. Many references from across the web provide an implicit equation for a single holed torus, such as equation (1) from Weisstein [9]. However, it is much more difficult to find references providing equations for 2 or 3 holed tori. Werewolftram provides an explicit implicit equation for a 3-holed torus [10]. Grimm and Hughes [4] provide an algorithm for parameterizing N -holed tori, however this method requires more work than “copying and pasting text into Mathematica/Maple; their method is also parametric and not implicit. Although there is a way to generate implicit N -holed tori on the Math Stack Exchange [8], this method does not give any information on how to determine the plotting bounds or how to determine their r parameter; a similar algorithm was referenced in [7] but yet again lacks plotting bounds for a given genus. Alternatively, the algorithm referenced in [7] could be utilized using a polar plot of the cosine function, however that is not what we have studied here; a brief overview of this thought is in section 2.1. The lack of plotting bounds for a given genus becomes problematic when trying to construct tori with a number of holes greater than 2.

The method I used to quickly generate implicit N -holed tori utilizes complex analysis, specifically complex roots of the equation: $z^N = 1$, $z \in \mathbb{C}$, $n \in \mathbb{N}$ and equations like it. \mathbb{N} is the set of natural numbers, and \mathbb{C} is the set of complex numbers. A reference to complex roots can be found in Brown and Churchill [2]. To my knowledge using complex roots to generate N -holed tori through symbolic software tools is new in the literature. I will now proceed to give an exposition on some of the geometric/topological terminology used in the paper.

A torus is the mathematical word for a doughnut. Topologically we can characterize two dimensional tori by the number of holes that they have. This number is called the genus g . The genus is related to the Euler characteristic χ as follows:

Definition 1 The Euler Characteristic of a compact 2-dimensional manifold is defined in terms of its genus g by:

$$\chi = 2 - 2g \tag{1}$$

Usually when we refer to a torus, we mean two-dimensional manifold of genus 1, see Figure 1.



Figure 1: A 1-Holed Torus, $g = 1$

Figure 1 is the usual shape students have in mind when discussing a torus. However, we can also have tori with more than one hole; a two holed torus is shown in Figure 2.

¹ Utah State University

² Work supported by a DoD SMART Scholarship



Figure 2: A 2-Holed Torus, $g = 2$

I developed an algorithm that easily generates N -holed tori using symbolic software tools to calculate complex roots of the equation $z^g = g^g$. The idea that we can make a torus of any genus comes from Werewolfram [10]. The idea on how to generate this equation quickly using software to solve, I believe, comes newly from this paper.

The idea on how to generate N-holed tori

Building off of the geometrical Idea in [10], all we needed was a way to quickly generate the holes. I decided to use complex roots of the equation:

Definition 2 Here we define a complex valued equation with variable z and parameter g the genus.

$$z^g = g^g \tag{2}$$

Originally, I began with the equation: $z^g = 1$, but then found that this equation did not space the holes of the torus far enough apart when plotting. From complex analysis, I knew that the roots plotted in the complex plane would have a discrete point symmetry, and thus look aesthetically pleasing. The construction of the implicit equation for N -holed tori is next.

First, by the fundamental theorem of algebra, we are guaranteed to have g roots ($1 \leq g < \infty$) over C to the above equation: $\{z_1, z_2, \dots, z_g\}, z \in C$. We solve the equation given in definition (2) using a solver in either Maple or Mathematica. Next, we break each of these z_i 's ($i \in N$, some natural number) into corresponding real points by taking the real and imaginary parts:

$$\{(\Re(z_1), \Im(z_1)), (\Re(z_2), \Im(z_2)), \dots, (\Re(z_g), \Im(z_g))\}$$

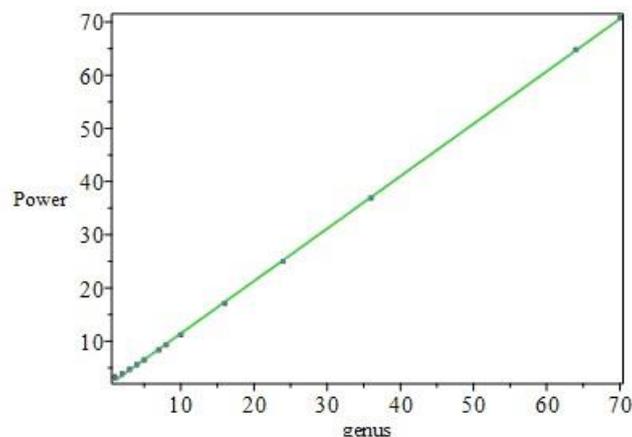
For ease and intuition, we define each of decomposed complex numbers by the following ordered pairs: $\{(x_1, y_1), (x_2, y_2), \dots, (x_g, y_g)\}$, such that $\Re(z_i) = x_i, \Im(z_i) = y_i$, etc. Then by using the idea from Werewolfram [10] we build an equation using these roots:

Definition 3 Here we define the implicit equation which determines the 2D surface (our N -holed tori) in Euclidean space.

$$\prod_{i=1}^g \left((x - x_i)^2 + (y - y_i)^2 - 1 \right) - 0.001 (x^2 + y^2 + 7z^2)^p = 0 \tag{3}$$

Where the power $p \in R^+$, is some positive real number. This equation will then create a torus with g holes for an appropriate choice of p ; finding a viable choice for p initially seemed like guess work. However, after some testing I produced a graph that cleanly describes what value p should be for a given g in Figure 3. A value of R^2 is given reflecting how good of a fit this equation is to the data I collected by experimenting with what values produced a compact g -holed tori.

Error plot for relation between genus and power



$$R^2 = 0.999871476177493$$

$$p = 1.59946890041590 + 0.985507617178373 g$$

Figure 3: A plot describing the relationship between the genus g , and the power p . The values on the line should give a good plot. The line is the equation $p \approx 1.599 + 0.986g$, the dots are the data points, and the error bars are small here, so they are not visible.

The bounds, b , of the plot are also important to know when plotting these tori in Maple and Mathematica; b is described just below. I also made a plot of what the bounds should be for a given genus g . In Mathematica, the bounds b would be reflected as the numbers that go into the three sets in the command ContourPlot3D as: $\{x, -b, b\}, \{y, -b, b\}, \{z, -b, b\}$, and the view range as: $PlotRange \rightarrow \{-b, b\}, \{-b, b\}, \{-b, b\}$

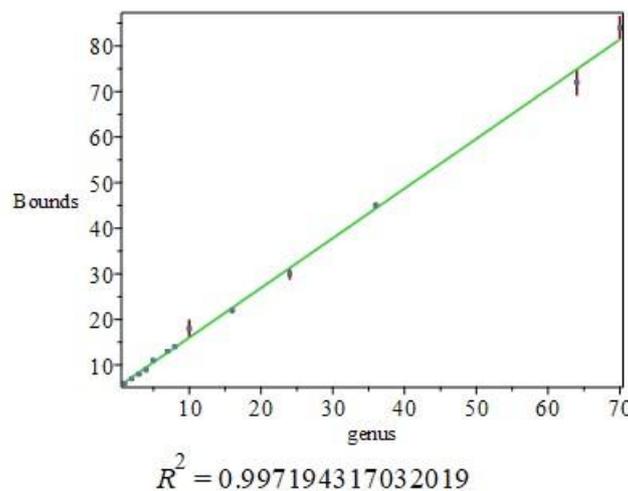
In Maple, the bounds b are given by: $x = -b..b, y = -b..b, z = -b..b$, and the view range as: $view = [-b..b, -b..b, -b..b]$. Figure 4 is like that of Figure 3, but is for the bounds.

Last, it is important to mention that by solving the equation $z^g = g^g$ we have in a sense made a g -gon in the complex plane. Connecting the neighboring points defined by the roots to equation 2 will define a g -gon in the plane; this is a result from Brown & Churchill [2]. Because the roots define a g -gon, we instantly have a discrete rotational symmetry that leaves the g -gon invariant.

An example given by $z^3 = 27$, ($g = 3$). This equation has a discrete rotational symmetry. When the triangle defined by connecting the roots is rotated by $120^\circ = \frac{2\pi}{3}$ the roots defining the triangle are invariant. The roots written as $x+iy \equiv (x,y)$, for the triangle are: $[(3, 0), (-\frac{3}{2}, -\frac{3}{2}\sqrt{3}), (-\frac{3}{2}, \frac{3}{2}\sqrt{3})]$.

Performing a rotation given by: $x = x \cos(\frac{2\pi}{3}) + iy \sin(\frac{2\pi}{3})$, $y = -x \sin(\frac{2\pi}{3}) + iy \cos(\frac{2\pi}{3})$ we get

Error plot for relation between genus and bounds



$$b = 5.10991167811580 + 1.09028459273798 g$$

Figure 4: A plot describing the relationship between the genus g , and the bounds for the plot b in the equation above to give a good plot. The line is the equation $b = \dots$, the dots are the data points, and the error bars are vertical from the points. the points: $[(-\frac{3}{2}, -\frac{3}{2}\sqrt{3}), (-\frac{3}{2}, \frac{3}{2}\sqrt{3}), (3, 0)]$ which permuted the list, but the set of elements is still the same. From an aesthetic perspective the author believes this symmetry makes the resulting plot more beautiful.

Using polar coordinates to generate tori and other algorithms

In this section we will give a brief overview on how to use the algorithm presented by [7] and [8] to make tori albeit without the bounds given. To make a N -holed tori following reference [8]’s approach, we go through the following steps.

Definition 4: Here is the algorithm presented by [8] which defines an n -holed torus in 3D space. It has 4 steps.

1. Step 1, Reference [8] presents function 4. Construct this function.

$$f(x) = \prod_{i=1}^n (x - (i-1))(x - i) = 0 \tag{4}$$

2. Step 2, Construct function 5

$$g(x,y) = f(x) + y^2 \tag{5}$$

3. Step 3, Construct function 6

$$h(x,y,z) = g(x,y)^2 + z^2 - r^2 \tag{6}$$

4. Step 4, Determine r for a given f , and plot.

Step 4 is left out of the algorithm, but could hopefully be implemented with information similar to that given by 4 and 3. In several ways the r parameter is similar to our p parameter.

Similarly, to reference [8], reference [7] takes a similar approach, although it includes more pedagogical variables. We will follow that algorithm here. It should be noted that most of the following algorithm was presented by [7], with whom I do not have a contact for.

Definition 5 Here we explain the algorithm of [7].

1. Step 1, Let: $f(x,y) = 0$ be an implicit curve in \mathbb{R}^2 with g holes.
2. Step 2, Then: $f(x,y)^2 + z^2 - \epsilon^2 = 0$, $\epsilon \in \mathbb{R}-0$ is an implicit surface of genus g ; epsilon is any sufficiently small number. The idea in step 2 is, that the implicit curve is expanded to a tubular shape. E.g.

$$f(x,y) = x^2 + y^2 - 1 \tag{7}$$

$$g(x,y) = ((x - 1)^2 + y^2)((x + 1)^2 + y^2) - 1 \tag{8}$$

Next, we give an example of this algorithm. In 7 $f(x,y)$ Gives a circle, and the tube would be $f(x,y)^2 + z^2 - \epsilon^2 = 0$, a torus. Additionally, in 7 $g(x,y)$ gives the Cassini ovals, thus $g(x,y)^2 + z^2 - \epsilon^2 = 0$, a double torus, respectively a surface of genus 2. However again, this algorithm would have trouble with a higher count of holes similar to [8] because of the bounds.

We can however utilize both of these if we were to have the bounds. Take for instance the polar equation with variables (r, θ) given in 9.

$$r^2 = \cos(g\theta) \tag{9}$$

$$(r^2 - \cos(g\theta))^2 + z^2 - \epsilon = 0 \tag{10}$$

However, experimenting with equation 9 with several epsilon values, the best picture the author was able to create with $\epsilon = 0.1$ was still not aesthetically pleasing. Figure 5 shows the output below in cylindrical coordinates with 10000 points. As the genus got larger this algorithm also broke down and the picture was unplottable with the computational power available. Note the jagged edges.



Figure 5: A 3-Holed Cosine Torus in Maple

Generating N -holed tori in Maple

In this section we will illustrate how to plot N -holed tori in Maple [5] through the use of two examples: a five holed torus, and a 64 holed torus. In the code, to name some variables again for clarity, we have the: genus g , bounds b , power p , and others which will be talked about as we go through the code. There are a few modifications that you could make to have more control over the torus as well. If you make g^s in the right hand side of equation 12 larger then the holes will be spaced further apart; likewise if the right hand side of equation 12 smaller the holes will be closer together. The g^s piece you can think of as the “radius” where the center of each of the holes in the tori are located; You can also think of it as controlling how far the holes are from the center of the torus. The code is given below, and each of the equations in this section, unless otherwise notes, can be directly copied and pasted into Maple to run the code. An algorithm is given below.

1. Step 1, We begin by defining the genus g in Maple.

$$g := 5 : \tag{11}$$

Then the next step is to set up and solve equation 2 for complex roots. In Maple we can use the command `solve` to find the complex roots of the equation; this is seen in equation 12 2.

2. Step 2, Solve equation 2 for z .

$$sol5 := [solve(z^g = g^s, z)] : \tag{12}$$

The next part of the tori generation algorithm is to break the complex roots into their real and imaginary parts through *Re* and *Im*. Then we make a sequence of the roots using *seq* which we will evaluate equation 3 on. This is done in equations 13 and 14.

3. Step 3, Get the real and imaginary parts of *z*.

$$\begin{aligned} myseq5Re &:= \{seq(x||i=Re(sol5[i]),i=1..g)\} : & (13) \\ myseq5Im &:= \{seq(y||i=Im(sol5[i]),i=1..g)\} : & (14) \end{aligned}$$

Next we union these sets together to make a whole set to evaluate equation 3 on. This is done in equation 15.

4. Step 4, Union the sets of solutions together.

$$myseq5_1 := myseq5Re \text{ union } myseq5Im : \quad (15)$$

In the next step the Maple command *evalf* converts the symbolic values solved for in equation 12 to numerical values. I have found that this makes the plotting slightly faster. For large *g*, ex. $g \gtrsim 70$ the plotting becomes time expensive on my computer.

5. Step 5, Convert from symbolic values to computer double values (Recall: float, int, double, string, etc. are examples of values a computer can have)

$$myseq5 := evalf(myseq5_1) : \quad (16)$$

Here is the polynomial we had previously in equation 3 represented in Maple by equation 17.

6. Step 6, Create the equation in our second algorithm. See equation 3.

$$mymul5 := mul\left((x + x||i)^2 + (y + y||i)^2 - 1, i = 1..g\right) : \quad (17)$$

To get the full equation we need to evaluate equation 17 on the set given by equation 16. This is done with the *eval* command in equation 18.

7. Step 7, Evaluate the Π equation on the set.

$$myexpression5 := eval(mymul5, myseq5) : \quad (18)$$

Here we define the bounds *b*, and power *p* to make the torus picture that we want.

8. Step 8, Define the proper bounds and power determined by 4 and 3.

$$b := 11 : \quad (19) \quad p := 6.45 : \quad (20)$$

In Maple you can change the number of points to make the plot prettier. I set the number of points to 400000 for these computations in Maple. This is shown in equation 21 9.

9. Step 9, Define the number of points that you want.

$$numpts := 400000 : \quad (21)$$

Last, all we need to do is plot the resulting implicit equation which is done with *implicitplot3d*. This is equation 22.

10. Step 10, Last, we plot the torus.

$$implicitplot3d\ myexpression5 - 0.001 * (x^2 + y^2 + 7 * z^2)^p, (22) \ x = -b..b, y = -b..b, z = -b..b, numpoints = numpts, style = surface, axes = none, view = [-b..b, -b..b, -b..b]$$

The result is shown in Figure 6.



Figure 6: A 5-Holed Torus in Maple

To show the full extent of the algorithm, we can even make Tori with many holes. To illustrate the power of the algorithm, Figure 7 is a 64-hole torus. To get the 64-Holed Torus in Figure 7, all you need to do is repeat the above code with the following swaps: $g := 64$;, $bounds := 72$;, $power := 64.79$;, $numpts := 400000$;.

These computations were performed in Maple 2016. There is 3D printing capability in Maple 2016, but the author was unable to find any command to change the extrusion thickness in Maple; changing the extrusion thickness may be an option in future releases of Maple, but likewise the author is unaware of these. Next, we will move onto how to do the same procedure in Mathematica.

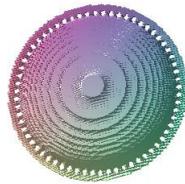


Figure 7: A 64-Holed Torus in Maple

Generating N -holed tori in Mathematica

We can also illustrate the N -holed tori generation algorithm in Mathematica [6]. I decided to illustrate the algorithm with the construction of a 7-holed torus. We first begin by defining the genus g , the power p , and the bounds b similar to what we did in Maple. The code follows the same as it did in Maple other than syntactical differences. Explicitly the code is given below in 23-34 with the output picture provided in Figure 7. A good reference on how to work with differential geometry in Mathematica is Gray et. all [3]; this reference was quite helpful for understanding the inner workings of Mathematica. We will begin by defining the genus g , power p , and bounds b .

1. Step 1, define the genus, power, and bounds.

$$g := 7 \quad (23)$$

$$p := 8.34 \quad (24)$$

$$b := 13.5 \quad (25)$$

Next, we solve equation 2 for the roots using Mathematica in equation 26.

2. Step 2, Solve the complex genus equation 2.

$$sol7 := Solve[z^g == g^g, z] \quad (26)$$

Similar to in Maple, we now break apart these complex roots to form a list of the real and imaginary parts of the roots. These parts are generated by the operations Re and Im ; this is done in equations 27-30.

3. Step 3, Get the real and imaginary parts of the solution.

$$myseq7Re1 := Re[z.sol7] \quad (27)$$

$$myseq7Re := Table[x[i] -> myseq7Re1[[i]], {i, 1, g}] \quad (28)$$

$$myseq7Im1 := Im[z.sol7] \quad (29)$$

$$myseq7Im := Table[y [i] -> myseq7Re1[[i]], {i, 1, g}] \quad (30)$$

Now we union these lists together to form a list of ordered pairs which will serve "make the holes" of the torus. We do this in equation 31.

4. Step 4, Union the solution sets together.

$$myseq7 := Union[myseq7Re,myseq7Im] \tag{31}$$

Taking the product as in equation 3 using the ordered pairs in equation 31 gives us equation 32. Taking it one step further, in equation 33 we evaluate the symbolic values generated by the Mathematica solver as floating-point values; this helps to speed up the computation for larger hole tori.

5. Step 5, plug the values into the Π equation 3.

$$A0=(x+x[i])^2+(y+y[i])^2 - 1 \tag{32}$$

$$mymul7 := Product[A0,\{i,1,g\}]/.myseq7 \tag{33}$$

$$myGenusSeq= mymul7/N \tag{34}$$

Lastly, we run the command *ContourPlot3D* as in 34 to actually plot the torus. This finishes the algorithm and we have our torus! A picture is provided of the result in Figure 8.

6. Step 6, Plot the n-holed torus.

$$A1 = myGenusSeq - 0.001 (x^2 + y^2 + 7z^2)^p == 0$$

$$A2 = \{x, -b, b\}, \{y, -b, b\}, \{z, -b, b\}$$

$$A3 = Axes -> None, PlotPoints -> 60$$

$$A4 = Mesh -> None, Boxed -> False$$

$$A5 = PlotRange -> \{\{-b, b\}, \{-b, b\}, \{-b, b\}\}$$

$$ContourPlot3D [A1, A2, A3, A4, A5] \tag{34}$$

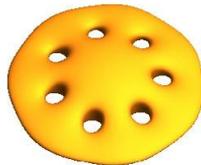


Figure 8: A 7-Holed Torus in Mathematica

3D Printing the torus using Mathematica

We can also export this torus to a .stl file and print it in Mathematica by calling the above code with the addition of *Extrusion -> 0.3*. The extrusion command is added so that the thickness of the 3D printed object helps hold it together structurally; this is also important for open surfaces, or surfaces that are not closed/compact. Then to get the .stl file for the 3D printing, call *Export["7Torus.stl", %]*, and the output will be the 7Torus.stl file in a directory on your computer; for my computer the file showed up in Documents. Alternatively, you can also call the command *Printout3D[myPlot, "7Torus.stl"]* when you set *myPlot = ContourPlot3D[...]*. *Printout3D* may take longer, but it also cleans up the plot more than *Export[...stl]* will. With *Printout3D* the resulting .stl file will be more "clean." Either option is viable for 3D printing, and I have given an example in 9 of a 7-Holed Torus I printed out.

Summary

Altogether, we have constructed a symbolic algorithm to generate implicit *N*-Holed tori in Maple and Mathematica through the use of equation 3, discussed the geometrical background of the torus,



Figure 9: A 3D Printing of a 7-Holed Torus

gave an example of 3D printing and given estimation plots of good values for plotting these tori in each of the 2 softwares. The author hopes that this will be helpful to students who are learning symbolic software, entrepreneurs of 3D printing, to artists for the symmetry in 3D modeling, to math students who would like to test topological algorithms on implicit equations, and to anyone interested in general.

Financial Support

This work was supported in part by a U.S. Department of Defense: Science, Mathematics, and Research for Transformation (DoD SMART) scholarship, for which the author is grateful.

Acknowledgements

The author thanks Dr. Charles Torre and Dr. Dariusz Wilczynski for their help with navigating the publication process. The author also thanks the reviewers who gave several helpful critiques and suggestions for the present work.

About the author:

Joshua J. Leiter is a PhD. physics graduate from Utah State University, and a Department of Defense SMART scholar who is interested in Wormholes. For his undergraduate degree he attended the University of Mount Union and graduated with Honors in both Mathematics (Applied) and Physics. Additionally, he greatly enjoys the topics of: differential geometry, differential equations, Lie symmetries, general relativity, radiation, multilinear algebra, 3D printing, and variational principles.

Joshua J. Leiter
joshualeiter@rocketmail.com

References

- [1] Bloomenthal J. *Introduction to Implicit Surfaces*. Morgan Kaufmann, 1997.
- [2] Brown J.W., and Churchill V.R. *Complex Variables and Applications*. McGraw-Hill Education, 2014.
- [3] Gray A. et al. *Modern Differential Geometry of Curves and Surfaces with Mathematica*. Chapman & Hall/CRC, 2018.
- [4] Grimm C., and Hughes J. Parameterizing N-holed Tori.
<https://web.engr.oregonstate.edu/~grimmc/content/papers/ms2003pt/ms2003pt.pdf>.
accessed February 1, 2021.
- [5] Maplesoft, Waterloo Maple Inc. Maple 2016. Waterloo, Ontario.
- [6] Wolfram Research, Inc. Mathematica, Version 11.3. Champaign, IL, 2018.
- [7] Computer Aided Geometric Design Reviewer. Reviewer comments. private communication.
- [8] Martins R.M., and User856 et al.
Implicit equation for a “double torus” (genus 2 orientable surface).
<https://math.stackexchange.com/questions/152256/implicit-equation-for-double-torus-genus-2-orientable-surface>.
See answer by User856, and comment by Leo below the response.
- [9] Weisstein E. W. "Torus".
<https://mathworld.wolfram.com/Torus.html>. From MathWorld-A Wolfram Web Resource.
- [10] Werewolf. Today I Learned: How to Plot a Doughnut.
<http://metamathological.blogspot.com/2013/01/today-i-learned-how-to-plot-doughnut.html?m=1>.
accessed October 6, 2020.