On Vector-valued Littlewood-Paley Theorem

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Abstract

In this paper we prove the existence of the Banach space-valued Littlewood–Paley theorem implies that a Banach space is isomorphic to a Hilbert space.

Keywords: Vector-Valued, Space-Valued, Hilbert Space, Banach Space

Introduction

Suppose that a function \( \psi \) is in \( S(R^n) \) with \( \text{supp } \psi \subset \left\{ \xi \in R^2 : \frac{1}{2} \leq |\xi| \leq 2 \right\} \) and \( |\psi(\xi)| \geq c > 0 \) if \( \frac{3}{5} \leq |\xi| \leq \frac{5}{3} \). Then one form of the classical Littlewood–Paley theorem on \( R^n \) says

\[
C \left\| f \right\|_{p} \leq \left\| \sum_{k=-\infty}^{\infty} \psi_k \ast f \left\| p \right\| \leq C \left\| f \right\|_{p} \tag{1}
\]

where \( 1 < p < \infty \), \( \psi_k (x) = 2^{kn} \psi \left( 2^k x \right) \), and \( c, C \) are constants independent of \( f \).

We study the vector-valued Littlewood-Paley theorem. To be precise, let \( B \) be a Banach space and \( L^p_B \left( R^n \right) \) be the space of strongly measurable \( B \)-valued function \( f \) for which \( \left\| f \right\|_{B} \in L^p \left( R^n \right) \). It is well known that if \( B \) is a Hilbert space, then the classical Littlewood-Paley theorem still holds

\[
C \left\| f \right\|_{L^p_B} \leq \left\| \sum_{k=-\infty}^{\infty} \psi_k \ast f \left\| p \right\| \leq C \left\| f \right\|_{L^p_B} \tag{2}
\]

where \( 1 < p < \infty \) and \( \psi \) is the same function as in (1).

We first prove that if \( B \) is a Banach space on (2) for one function \( \psi \) mentioned above, then (2) holds for a more general family of operators.

Definition (1.1) [3]:

A family of operators \( \{S_k\}_{k \in \mathbb{Z}} \) is said to be an approximation to the identity if for \( 0 < \varepsilon \leq 1 \) and \( \delta = \varepsilon - \varepsilon' > 0 \) there is a constant \( C \) such that for all \( k \in \mathbb{Z} \) and all \( x, x', y \) and \( y' \in R^n, \) the kernels of \( S_k \), satisfy the following conditions:

(i) \( |S_k(x, y)\leq C \left( \frac{2^{-k\varepsilon}}{2^{-k}+|x-y|} \right)^{2\varepsilon} \)

(ii) \( |S_k(x, y) - S_k(x', y)| \leq C \left( \frac{|x-x'|}{2^{-k}+|x-y|} \right)^{\varepsilon} \left( \frac{2^{-k\varepsilon}}{2^{-k}+|x-y|} \right)^{2\varepsilon} \) for \( |x-x'| \leq \frac{1}{2} \left( 2^{-k} + |x-y| \right) \)
\( iii \) \( |S_k(x, y) - S_k(x, y')| \leq C \left( \frac{|y - y'|^{2^{-k}}}{(2^{-k} + |x - y|)^{n+\delta}} \right) \) for \( |y - y'| \leq \frac{1}{2} (2^{-k} + |x - y|) \)

\( iv \) \( \left| S_k(x, y) - S_k(x, y') - [S_k(x', y) - S_k(x', y')] \right| \leq C \left( \frac{|x - x'|^{2^{-k}}}{(2^{-k} + |x - y|)^{n+\delta}} \right) \) for \( |x - x'| \leq \frac{1}{2} (2^{-k} + |x - k|) \)

\( \text{and } |y - y'| \leq \frac{1}{2} (2^{-k} + |x - y|) \), and \( \delta = \varepsilon - \varepsilon' > 0 \)

\( v \) \( \int S_k(x, y) \, dy = \int S_k(x, y) \, dx = 1 \) for all \( k \in \mathbb{N} \).

All of the conditions (i)–(v) on the approximate identities are needed for the Calderon reproducing formula.

**Definition (1.2)** \[3\]:

Fix two exponents \( 0 < \beta \leq 1 \) and \( \gamma > 0 \). A \( B \)-valued function \( f \), where \( B \) is a Banach space, is said to be a test function of type \( (\beta, \gamma) \) centered at \( x_0 \in \mathbb{R}^n \) with width \( d > 0 \) if \( f \) satisfies the following conditions:

\( i \) \( \left| f\left( x \right) \right|_B \leq C \frac{d^\gamma}{\left( d + |x - x_0| \right)^{n\gamma}} \),

\( ii \) \( \left| f\left( x \right) - f\left( x' \right) \right|_B \leq C \frac{\left( |x - x'| \right)^\beta}{\left( d + |x - x_0| \right)^{n\gamma}} \) for \( |x - x'| \geq \frac{1}{2} (d + |x - x_0|) \),

\( iii \) \( \int_{\mathbb{R}^n} f\left( x \right) \, dx = 0 \)

The collection of all test functions of type \( (\beta, \gamma) \) centered at with width \( d > 0 \) will be denoted by \( M_{\beta, \gamma}^{(\beta, \gamma)}(x_0, d) \). If \( f \in M_{\beta, \gamma}^{(\beta, \gamma)}(x_0, d) \) the norm of \( f \) in \( f \in M_{\beta, \gamma}^{(\beta, \gamma)}(x_0, d) \) is defined by

\[ \left\| f \right\|_{M_{\beta, \gamma}^{(\beta, \gamma)}(x_0, d)} = \inf \{ C \geq 0 \} \]

If (i),(ii),(iii) of Definition (1.2) hold, we denote the class of all \( f \in M_{\beta, \gamma}^{(\beta, \gamma)}(0, 1) \) by \( M_{\beta, \gamma}^{(\beta, \gamma)} \). It is easy to see that \( M_{\beta, \gamma}^{(\beta, \gamma)} \) is a Banach space under the norm \( f \in M_{\beta, \gamma}^{(\beta, \gamma)} < \infty \). It is also easy to see that \( M_{\beta, \gamma}^{(\beta, \gamma)} = f \in M_{\beta, \gamma}^{(\beta, \gamma)}(x_0, d) \) for \( x_0 \in \mathbb{R}^n \) and \( d > 0 \), with equivalent norms.

**Theorem (1.3)** \[3\]:

Suppose that \( \{ S_k \} \) is approximation to the identity defined in (4) below. Set \( D_k = S_k - S_{k-1} \). Then there exists a family of operators \( \{ \tilde{D}_k \}_{k \in \mathbb{N}} \) such that for all \( f \in M_{\beta, \gamma}^{(\beta, \gamma)} \),

\[ f = \sum_{k \in \mathbb{N}} \tilde{D}_k D_k (f) \quad (3) \]

where the series converges in the norm of \( M_{\beta, \gamma}^{(\beta, \gamma)} \) with \( \beta' < \beta \) and \( \gamma' < \gamma \). Moreover, \( \tilde{D}_k (x, y) \) the kernel of \( \tilde{D}_k \), satisfy the following estimates: for \( \varepsilon', 0 < \varepsilon' < \varepsilon \) where \( \varepsilon \) is the regularity exponent of \( S_k \), there exists a constant \( C > 0 \).

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Such that

(i) \( |\tilde{D}_k (x, y)| \leq C \frac{2^{-k\varepsilon'}}{(2^{-k} + |x - y|)^{n\varepsilon'}} \)

(ii) \( |\tilde{D}_k (x, y) - \tilde{D}_k (x', y)| \leq C \left( \frac{|x - x'|}{2^{-k} + |x - y|} \right)^{\varepsilon'} \frac{2^{-k\varepsilon'}}{(2^{-k} + |x - y|)^{n\varepsilon'}} \) for \( |x - x'| \leq \frac{1}{2} (2^{-k} + |x - y|) \)

(iii) \( \int \tilde{D}_k (x, y) dy = \int \tilde{D}_k (x, y) dx = 0 \) for all \( k \in \mathbb{Z} \).

Since \( \sum_{k=\pm} D_k (f) = f \) in the strong topology of \( L^p_b \left( \mathbb{R}^n \right) \), it is easy to see that \( M^{(\beta, \gamma)}_b \) is dense in \( L^p_b \left( \mathbb{R}^n \right) \) for all \( 0 < \beta \leq 1 \) and \( \gamma > 0 \).

**Theorem (1.4) [3]:**

Suppose that \( B \) is a Banach space. Suppose \( \{S_k\} \) is an approximation to the identity and \( D_k = S_k - S_{k-1} \), and the Littlewood-Paley theorem holds for \( \{D_k\} \), that is, for \( 1 < p < \infty \),

\[
\|f\|_{L^p_B} \leq C \left\| \left\{ \sum_{k=1}^{\infty} |D_k (f)|^2 \right\}_k \right\|_{L^p_B} \leq C \|f\|_{L^p_B}
\]

(4)

Then the Littlewood-Paley theorem holds for \( \{E_k\} \) where \( E_k = R_k - R_{k-1} \) and \( \{R_k\} \) is an approximation to the identity, that is, for \( 1 < p < \infty \)

\[
\|f\|_{L^p_B} \leq C \left\| \left\{ \sum_{k=1}^{\infty} |E_k (f)|^2 \right\}_k \right\|_{L^p_B} \leq C \|f\|_{L^p_B}
\]

(5)

**Proof:**

We need to show (5) for all \( f \in M^{(\beta, \gamma)}_b \). Suppose that (5) holds and \( E_k = R_k - R_{k-1} \) where \( \{R_k\} \) is an approximation to the identity. By Theorem (1.3) for all \( f \in M^{(\beta, \gamma)}_b \) with \( 0 < \beta \leq 1 \) and \( \gamma > 0 \), we have

\[
E_k (f) = \sum_{j \in \mathbb{Z}} E_k \tilde{D}_j D_j (f)
\]

It is easy to check that \( E_k \tilde{D}_j (x, y) \), the kernel of \( E_k \tilde{D}_j \), satisfies the following estimates

\[
|E_k \tilde{D}_j (x, y)| \leq C 2^{-k\varepsilon''} \frac{2^{-2(k-j)\varepsilon'}}{(2^{-2(k-j)} + |x - y|)^{n\varepsilon'}}
\]

(6)

where \( 0 < \varepsilon'' < \varepsilon' < \varepsilon \) and \( a \wedge b \) denotes the minimum of \( a \) and \( b \)

Hence

\[
\left\| \left\{ \sum_{k \in \mathbb{Z}} |E_k (f)|^2 \right\}_p \right\| \leq C \left\| \left\{ \sum_{k \in \mathbb{Z}} \left[ \sum_{j \in \mathbb{Z}} |M (D_j (f))|^2 \right]^{1/2} \right\}_p \right\| \leq C \left\| \left\{ \sum_{j \in \mathbb{Z}} |M (D_j (f))|^2 \right\}_p \right\|^{1/2}
\]
\[
\leq C \left\| \left\{ \sum_{j \in \mathbb{Z}} D_j(f)^2 \right\}^{1/2} \right\|_p
\]  

(7)

Where \( M \) is the Hardy-Littlewood maximal function; the last inequality follows from the Fefferman-Stein vector-valued maximal inequality [1].

The proof of the inverse inequality of (7) is the same, and hence, this completes the proof.

**Corollary (1.5)[2]:** Consider the assumption of Theorem (1.4) then

\[
C \sum_{j=1}^n \|f_j\|_{L^p_B} \leq \left\| \left\{ \sum_{k=1}^n \sum_{\xi \in \mathbb{Z}} |D_k(f_j)|^2 \right\}^{1/2} \right\|_p \leq C \sum_{j=1}^n \|f_j\|_{L^p_B},
\]

and

\[
C \sum_{j=1}^n \|f_j\|_{L^p_B} \leq \left\| \left\{ \sum_{k=1}^n \sum_{\xi \in \mathbb{Z}} |E_k(f_j)|^2 \right\}^{1/2} \right\|_p \leq C \sum_{j=1}^n \|f_j\|_{L^p_B},
\]

for all \( f_j \in M_B(\beta, r) \).

**Proof:** for \( 0 < \beta < 1 \) and \( r > 0 \) we have

\[
\sum_{j=1}^\infty E_k(f_j) = \sum_{j=1}^\infty E_k \overline{D}_i D_i(f_j)
\]

\( E_k \overline{D}_i (x, y) \) is the kernel of \( E_k \overline{D}_i \) by Theorem (14). Satisfying the estimates of (6) in Theorem (1.4). Then we have

\[
\left\| \left\{ \sum_{j=1}^\infty \sum_{k=1}^\infty |E_k(f_j)|^2 \right\}^{1/2} \right\|_p \leq C \left\| \left\{ \sum_{j=1}^\infty \sum_{k=1}^\infty \|E_k(f_j)\|_p \right\}^{1/2} \right\|_p
\]

**Theorem (1.6) [3]:**

Let \( B \) be any Banach space, \( n \geq 1 \), \( a_0 \in B, \ldots, a_n \in B \). Let \( 1 < \lambda_1 < \ldots < \lambda_n < \lambda_{n+1}, \ldots, \lambda_j \) be the integers for all \( j \) and

\[
2\pi \left\{ \frac{\lambda_1}{\lambda_2} + \frac{\lambda_1 + \lambda_2}{\lambda_3} + \ldots + \frac{\lambda_1 + \lambda_2 + \ldots + \lambda_n}{\lambda_{n+1}} \right\} \leq \alpha < 1
\]

(8)

Let \( F_n(\theta_1, \ldots, \theta_n) = a_0 + a_1 e^{i\theta_1} + \ldots + a_n e^{i\theta_n}, \quad 0 \leq \theta_k \leq 2\pi \), for \( 1 \leq k \leq n \) and \( F_n(i) = a_0 + a_1 e^{i\theta_1} + \ldots + a_n e^{i\theta_n}, \quad 0 \leq \theta \leq 2\pi \). Then

\[
(1 - \alpha)\|F_n\|_2 \leq \|F_n\|_2 \leq (1 + \alpha)\|F_n\|_2
\]

(9)

where, the \( L^2 \)–norm will always be the normalized \( L^2 \left[ 0, 2\pi \right] \) norm or \( L^2 \left[ 0, 2\pi \right] \) with respect to all the variables.

To prove this theorem, set

\[
F_{n,k}(t, \theta_{k+1}, \ldots, \theta_n) = a_0 + a_1 e^{i\theta_1} + \ldots + a_k e^{i\theta_k} + a_{k+1} e^{i\theta_{k+1}} + \ldots + a_n e^{i\theta_n}
\]

(10)

We will prove

\[
\left\| F_{n,k} - \varepsilon_k \right\|_2 \leq \left\| F_{n,k} \right\|_2 \leq \left\| F_{n,k-1} \right\|_2 + \varepsilon_k \left\| F_n \right\|_2
\]

(11)

where \( 1 \leq k \leq n \) and \( \varepsilon_k = 2\pi \frac{\lambda_1 + \ldots + \lambda_{k-1}}{\lambda_k} \).
Observe first that once (11) is proved we obviously obtain
\[
(1 - \varepsilon_1 - \ldots - \varepsilon_n) \|F_n\|_2 \leq \|F_{n,k}\|_2 \leq \|F_{n,k-1}\|_2 + \varepsilon_k \|F_n\|_2 \leq (1 + \varepsilon_1 + \ldots + \varepsilon_n) \|F_n\|_2
\]
which yields the theorem.

The second observation is that the inequality in (11) for \(1 \leq k < n\) follows from the inequality in (11) for \(k = n\). Indeed, let us freeze \(\theta_{k+1}, \ldots, \theta_n\) and write
\[
\tilde{a}_n = a_n + a_{k+1} e^{i\theta_{k+1}} + \ldots + a_n e^{i\theta_n}
\]
We now apply (11) with \(n\) being replaced by \(k\), and obtain
\[
\left\|\tilde{a}_n + a_k e^{i\theta_k} + \ldots + a_k e^{i\theta_k} \right\|_2 \leq \left\|a_n + a_k e^{i\theta_k} + \ldots + a_k e^{i\theta_k} \right\|_2
\]
Writing \(\phi\) for \((\theta_{k+1}, \ldots, \theta_n)\) and using symbolic notation, we have obtained
\[
A(\phi) - \varepsilon_k B(\phi) \leq C(\phi) \leq A(\phi) + \varepsilon_k B(\phi)
\]
Now we take \(L^2\) norms with respect to \(\phi\) and obtain
\[
\left\|A(\phi)\right\|_{L^2(\phi)} - \varepsilon_k \left\|B(\phi)\right\|_{L^2(\phi)} \leq \left\|C(\phi)\right\|_{L^2(\phi)}
\]
Here we use the following observation: \(f(x) \geq 0\), \(g(x) \geq 0\), \(h(x) \geq 0\), and \(h(x) \geq f(x) - g(x)\) imply
\[
\|f\|_2 \geq \|f\|_2 - \|g\|_2, \quad \text{since} \quad h(x) + g(x) \geq f(x) \quad \text{obviously implies} \quad \|h\|_2 + \|g\|_2 \geq \|f\|_2.
\]
Since \(\left\|A(\phi)\right\|_{L^2(\phi)} = \left\|F_{n,k-1}\right\|_2, \quad \left\|B(\phi)\right\|_{L^2(\phi)} = \left\|F_n\right\|_2\) and \(\left\|C(\phi)\right\|_{L^2(\phi)} = \left\|F_{n,k-1}\right\|_2\), the inequality of (11) with \(k = n\). Note first that
\[
f^\phi_n \left(t + \frac{2k\pi}{\lambda_n}\right) = f^{\phi\phi}_{n-1} \left(t + \frac{2k\pi}{\lambda_n}\right) + a_n e^{i\phi_{k-1}}
\]
We now introduce
\[
f^{\phi\phi\phi\phi}_n (t, k, s) = f^\phi_{n-1} \left(t + \frac{2k\pi}{\lambda_n} + \frac{2\pi s}{\lambda_n}\right) + a_n e^{i\phi_{k-1}}
\]
If \(0 \leq s \leq 1\). Then
\[
\left|f_n \left(t + \frac{2k\pi}{\lambda_n}\right) - f^{\phi\phi\phi\phi}_n (t, k, s) \right|_2 \leq \|a_1\|_B \frac{2\pi \lambda_1}{\lambda_n} + \ldots + \|a_{n-1}\|_B \frac{2\pi \lambda_{n-1}}{\lambda_n}
\]
\[
\left|f_n \left(t + \frac{2k\pi}{\lambda_n}\right) - f^{\phi\phi\phi\phi}_n (t, k, s) \right|_2 \leq \sup\|a_1, \ldots, a_{n-1}\|_B \varepsilon_n
\]
We obviously have
\[
\|a_k\|_B \leq \|F_n\|_2 \quad \text{for} \quad 1 \leq k \leq n
\]
Since \(a_k\) are the Fourier coefficients of \(F_n\). Therefore,
\[
\left|f_n \left(t + \frac{2k\pi}{\lambda_n}\right) - f^{\phi\phi\phi\phi}_n (t, k, s) \right|_2 \leq \varepsilon_n \|F_n\|_2
\]
Taking the \(L^2\) norm with respect to all the variables \(t, k \in \{0, 1, \ldots, \lambda_n\}\) and \(S \in [0, 1]\), we obtain
\[
\left\{ \frac{1}{\lambda_n} \sum_{k=0}^{\lambda_n-1} \left| f_n \left( t + \frac{2k\pi}{\lambda_n} \right) \right|^2 \right\}^{\frac{1}{2}} - \frac{1}{2\pi} \int_0^{2\pi} \int_0^{\lambda_n} \left| f_n^* (t, k, s) \right| ds \, dt \leq \epsilon_n \| F_n \|_2 \quad (16)
\]

But
\[
\int_0^{\lambda_n} \sum_{k=0}^{\lambda_n-1} \left| f_n^* (t, k, s) \right|^2 ds = \frac{1}{\lambda_n} \sum_{k=0}^{\lambda_n-1} \int_0^{\lambda_n} \left( t + \frac{2k\pi}{\lambda_n} + \frac{2\pi s}{\lambda_n} \right) + a_n e^{i\lambda s} \right|^2 ds = \frac{1}{2\pi} \int_0^{2\pi} \left| f_{n-1} (t + \theta) + a_n e^{i\lambda s} \right|^2 d\theta = \frac{1}{2\pi} \int_0^{2\pi} \left| f_{n-1} (\theta) + a_n e^{i\lambda s} \right|^2 d\theta \quad (17)
\]

Then (16) yields
\[
\left\{ \left[ \int_0^{2\pi} \left| f_n (t) \right|^2 dt \right] \left[ \int_0^{2\pi} \left| f_{n-1} (\theta) + a_n e^{i\lambda s} \right|^2 d\theta dt \right] \right\}^{\frac{1}{2}} \leq \epsilon_n \| f_n \|_2 \quad (18)
\]

Which is the required estimate.

**Theorem (1.7) [3]:**
Suppose that \( B \) is a Banach space. If the \( B \)-valued Littlewood-Paley theorem (4) holds for some \( 1 < p_0 < \infty \) and where \( D_k = S_k - S_{k-1} \) and \( \{ S_k \} \) is an approximation to the identity, then \( B \) is isomorphic to a Hilbert space.

**Proof:**
First observe that if (4) holds for some \( 1 < p_0 < \infty \), then (4) holds for all \( 1 < p < \infty \). We define the operator \( T \) on \( L^{p_0}_B (R^n) \) by \( T(f) = \{ D_k (f) \}_{k \in \mathbb{Z}} \). The fact that (4) holds for means that is a bounded operator from \( L^{p_0}_B (R^n) \) to \( L^{p_1}_B (R^n) \) where
\[
L^{p_0}_B (R^n) = \left\{ f_k (x) \in \left( \sum_{k \in \mathbb{Z}} \left| f_k (x) \right|^p \right)^{\frac{1}{p}} \right\}.
\]

Here we say that an operator \( T \) is a vector-valued Calderon-Zygmund operator if \( T \) is a continuous linear operator from \( L^{p_0}_B (R^n) \) to \( L^{p_1}_B (R^n) \) for some \( 1 < p_0 < \infty \) with the kernel \( k(x, y) \) mapping \( R^n \times R^n \) to the space of all bounded operators from \( B \) to \( L^2_B \) and satisfy the following conditions: for some \( \epsilon > 0 \), there is a constant \( C \geq 0 \) such that
\[
\| k(x, y) \| \leq C \| x - y \|^{n+\epsilon} \quad \forall x, y \in R^n \quad \text{with} \quad \| x \| \neq \| y \| \quad \text{for all} \quad \| x \| - \| x \| \leq \frac{1}{2} \| x - y \|, \quad (19)
\]
\[
\| k(x, y) - k(x', y) \| \leq C \| x - x' \|^{n+\epsilon} \| x - y \|^{n+\epsilon} \quad \text{for all} \quad \| x - x' \| \leq \frac{1}{2} \| x - y \|, \quad (20)
\]
\[
\| K(x, y) - K(x', y) \| \leq C \| x - x' \|^{n+\epsilon} \| x - y \|^{n+\epsilon}, \quad \text{for all} \quad \| x - x' \| \leq \frac{1}{2} \| x - y \|, \quad (21)
\]

By the Calderon-Zygmund real-variable theory, \( T \) also is bounded from \( L^{p_0}_B (R^n) \) to \( L^{p_1}_B (R^n) \) for all \( 1 < p < \infty \).
Let $\psi, \phi \in S(\mathbb{R}^n)$ with $\text{supp} \hat{\psi} \subset \left\{ \xi \in \mathbb{R}^n : \frac{1}{2} \leq |\xi| \leq 2 \right\}$, $|\hat{\psi}(\xi)| \geq c > 0$ if $\frac{3}{5} \leq |\xi| \leq \frac{5}{3}$, and $\text{supp} \hat{\phi} \subset \left\{ \xi \in \mathbb{R}^n : |\xi| \leq 1 \right\}$, $\sup_{x \in [0,2\pi]} |\phi(x)| \geq \delta > 0$. Suppose that we accept the $B$–valued Littlewood-Paley theorem in (4) for some $1 < p_0 < \infty$ and $\{D_k\}$ where $D_k = S_k - S_{k-1}$ and $\{S_k\}$ is an approximation to the identity. By Theorem (1.4) and the first observation above, we may assume the following inequalities hold:

$$c \left\| f \right\|^2_{L^p_B} \leq \sum_{k \in \mathbb{Z}} \left\| \psi_k \ast f \right\|^2_{L^p_B} \leq C \left\| f \right\|^2_{L^p_B} \tag{22}$$

where the constants $c$ and $C$ are independent of $f$.

Now consider the function $f(x) = f_n(x) \phi(x) = \left[ a_1 e^{ik_1} + \ldots + a_n e^{ik_n} \right] \phi(x)$ where $\lambda_j = 3^j$ for $1 \leq j \leq n$. Then (22) implies

$$\sum_{j=1}^n \left\| a_j \right\|^2_{L^p_B} \leq \left\| f_n \right\|^2_{L^p_B(0,2\pi)}$$

We now apply Theorem (1.6) and obtain

$$\left\| f_n \right\|^2_{L^p_B(0,2\pi)} \leq \left\| a_1 e^{i\epsilon_1} + \ldots + a_n e^{i\epsilon_n} \right\|^2_{L^p_B(0,2\pi)}$$

Now we have

$$\left\| a_1 e^{i\epsilon_1} + \ldots + a_n e^{i\epsilon_n} \right\|^2_{L^p_B(0,2\pi)} \leq 2^n \left\| \sum \epsilon_i a_i - \sum \epsilon_i a_i \right\|_B$$

where the series is extended over all sequences $\epsilon = (\epsilon_1, \ldots, \epsilon_n)$ with $\epsilon_k$ being independent Bernoulli random variables, that is, $\epsilon_k = \pm 1$ for $1 \leq k \leq n$.

This shows that for any $n \geq 1$ and $a_1, a_2, \ldots, a_n \in B$, there exist constants and such that

$$c \sum_{j=1}^n \left\| a_j \right\|^2_{L^p_B} \leq \frac{1}{2^n} \sum \left\| \epsilon_i a_i + \ldots + \epsilon_n a_n \right\|_B^2 \leq C \sum_{j=1}^n \left\| a_j \right\|_B$$

which implies that $B$ is isomorphic to a Hilbert space, and hence, Theorem (1.7) is proved.

**Results and Discussion**

Our main results is collary (1.5) which is a deduction of theorem (1.4) and depends on its main assumption : the set $\{S_k\}$ is an approximation to the identity.

**Conclusion and Recommendations**

Does theorem (1.4) holds for $1 \leq p \leq \infty$ ?

**References**


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