Global Existence and Exponential Stability for a TimoshenkoBeam Model

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Abstract

We prove the existence of solution for a Timoshenko beam model using Semigroups Theory, regularity results and a Theorem associated to the Lumer Phillips Theorem. Also, using multiplicative techniques and the classic Gearhart Theorem, introduced in Liu- Zheng (Liu, 1999), we prove that energy associated to the system decays exponentially to zero when $t \to +\infty$. *Stability for another Timoshenko beam model has been considered in(Raposo, et all, 2005).*

Keywords: Timoshenko beam, Exponential stability, Gearhart Theorem.

1. Introduction

Our main goal is to prove the exponential stability of the model known as Timoshenko beams in the case when the friction acts naturally in both the rotation angle of filaments and the transversal vibrations of the beam.
In 1921, Timoshenko (Timoshenko,1921) introduced the following coupled hyperbolic system

$$\begin{aligned} \rho u_{tt} &= [k(u_x - \psi)]_x \quad in \ \ ]0, L[\times]0, +\infty[\\ I_p \psi_{tt} &= (EI\psi_x)_x + k(u_x - \psi) \quad in \ \ ]0, L[\times]0, +\infty[\end{aligned} \tag{1.1}$$

that describes transversal vibrations of a beam without damping in the equilibrium state, where t is the time variable and x is the space coordinate along the beam of length L. The function $u = u(t, x)$ is the transversal displacement of the beam in the equilibrium state and $\psi = \psi(t, x)$ is the rotation angle of the beam. The coefficients ρ, I_p, E, I y k are respectively the density (the mass per unit length), the polar moment of inertia of a cross section, the Young's modulus of elasticity, the moment of inertia of a cross section and the shear modulus.

For a physical derivation of this system we cite (Graff, 1975).
System (1.1) together with boundary conditions of the form

$$EI\psi_x|_{x=0}^{x=L} = 0 \quad , \quad k(u_x - \psi)|_{x=0}^{x=L} = 0$$

is conservative, and so the total energy of the beam remains constant along the time.
Kim and Renardy (Kim,1987) considered the system (1.1) together with two boundary controls of the form

$$k\psi(L,t) - ku_x(L,t) = \alpha u_t(L,t), \quad EI\psi_x(L,t) = -\beta\psi_t(L,t), \forall t \geq 0.$$

and proved exponential decay for the total energy, using multiplicative techniques. They also provided numerical estimates to the eigenvalues of the operator associated to the system. A similar result was also established by Feng (Feng, et all, 1998).

Soufyane and Weybe (Soufyane, 2003) proved that is possible to stabilize uniformly (1.1) by using a unique locally distributed feedback of the form $b(x)\psi_t$ in the left hand side of the second equation in (1.1), where b is a positive and continuous function, which satisfies

$$b(x) \geq b_o > 0, \forall x \in [a_0, a_1] \subset [0, L]$$

and proved that uniform stability holds if and only if the wave speeds are equal, that is $\frac{k}{\rho} = \frac{EI}{I_p}$. Otherwise only the asymptotic stability has been proved.

Muñoz and Racke (Muñoz, 2007) obtained a similar result when the damping function $b = b(x)$ is allowed to change its sign.
Since $E(t)$ the total energy associated to Timoshenko beam system,

$$E(t) := \frac{1}{2}\int_0^L \{\rho_1 |u_t|^2 + \rho_2 |\psi_t|^2 + b|\psi_x|^2 + k|u_x - \psi|^2\} dx$$

have a non positive derivative, that is $E'(t) \leq 0$, the system is dissipative.
Then, we want to know if $E(t) \to 0$ when $t \to +\infty$, and what is its decay rate?. The answer is affirmative, that is, there exist positive constants C and γ such that,

$$E(t) \leq CE(0)e^{-\gamma t} \;\; \textit{for every } t > 0.$$

Thus, our main goal is to prove the existence and uniqueness of global solution of a Timoshenko beam system and its exponential stability.

We prove the existence of global solution for a Timoshenko beam system by using semigroups theory. Here, we give a full proof. Also, using multiplicative techniques and the classic Gearhart Theorem, introduced in Liu- Zheng (Liu, 1999), we prove that energy associated to the system decays exponentially to zero when $t \to +\infty$.

Analogously, we prove for the case Timoshenko beam model with double damping.
Remember that this analytic technique were applied to dissipative problems, like (Santiago, 2003, 2004, 2012).
Our paper is organized as follows. In section 2 we state the preliminary results that we will use. In section 3 we prove the existence and uniqueness of global solution.

In section 4, we prove the exponential decay of the solution.
In section 5, we prove the global existence and exponential stability for a Timoshenko beam model with double damping.

2.Preliminaries

To prove existence of solution of the Timoshenko beams system, we will use a result associated to Lumer Phillips Theorem. Here we state this important result. The proof can be seen in (Pazy, 1983).

Theorem 2.1 *Let A be a linear operator with domain $D(A)$ dense in a Hilbert Space. If A is dissipative and $0 \in \rho(A)$, then A is the infinitesimal generator of a C_o semigroup of contraction in this Hilbert space.*
By other hand, we know that the problem of providing an estimate to the energy $E(t)$ of the form

$$E(t) \leq CE(0)e^{-wt}, \forall t \geq 0,$$

is equivalent to providing exponential stability for semigroup $S(t)$

$$\|S(t)\| \leq Ce^{-wt}, \forall t \geq 0,$$

we cite (Liu, 1999).
A necessary and sufficient condition for a semigroup C_o to be exponentially stable is given by the following result
Theorem 2.2 (Gearhart) *Let $(S(t))_{t\geq 0}$ be a C_o semigroup of contraction in a Hilbert space. Then, $(S(t))_{t\geq 0}$ is exponentially stable (that is $\exists M \geq 1$, $\mu > 0$ such that $\|S(t)\| \leq Me^{-\mu t}$, $\forall t \geq 0$) if and only if*

a) $\rho(A) \supset iR := \{i\beta, \beta \in R\}$
b) $\limsup_{|\beta| \to \infty} \|(i\beta I - A)^{-1}\| < \infty$.

We will respectively use Theorems 2.1 and 2.2 to prove existence of solution and exponential stability of a Timoshenko beam system.

3. The Abstract Cauchy Problem and Existence of Solution

Here we study the following system type Timoshenko with a damping term

$$\rho_1 u_{tt} - k[u_x - \psi]_x + \alpha(x) u_t = 0, \quad (x,t) \in]0,L[\times R^+ \tag{3.1}$$

$$\rho_2 \psi_{tt} - b\psi_{xx} - k[u_x - \psi] = 0, \quad (x,t) \in]0,L[\times R^+ \tag{3.2}$$

$$u(0,t) = u(L,t) = 0, \quad \psi(0,t) = \psi(L,t) = 0, \ t \in R^+ \tag{3.3}$$

$$u(x,0) = u_o(x), \quad u_t(x,0) = u_1(x), \ x \in [0,L] \tag{3.4}$$

$$\psi(x,0) = \psi_o(x), \quad \psi_t(x,0) = \psi_1(x), \ x \in [0,L] \tag{3.5}$$

where L is the length of the beam, and ρ_1, ρ_2, b, k are positive constants. Here $\alpha(x)$ is a continuous function such that $\alpha(x) \geq a > 0$.

To get the energy associated to the system, multiply (3.1) by u_t and integrate on $[0,L]$, having

$$\frac{1}{2}\frac{\partial}{\partial t}\int_0^L \{\rho_1 |u_t|^2 + k|u_x|^2\}dx - k\int_0^L \psi u_{xt} dx + \int_0^L \alpha(x)|u_t|^2\, dx = 0, \tag{3.6}$$

also multiply (3.2) by ψ_t and integrate on $[0,L]$ to have

$$\frac{1}{2}\frac{\partial}{\partial t}\int_0^L \{\rho_2 |\psi_t|^2 + b|\psi_x|^2 + k|\psi|^2\}dx - k\int_0^L \psi_t u_x dx = 0. \tag{3.7}$$

Summing (3.6) with (3.7), we get

$$\frac{\partial}{\partial t}\frac{1}{2}\int_0^L \{\rho_1 |u_t|^2 + \rho_2 |\psi_t|^2 + b|\psi_x|^2 + k|u_x - \psi|^2\}dx + \int_0^L \alpha(x)|u_t|^2\, dx = 0. \tag{3.8}$$

Let

$$E(t) := \frac{1}{2}\int_0^L \{\rho_1 |u_t|^2 + \rho_2 |\psi_t|^2 + b|\psi_x|^2 + k|u_x - \psi|^2\}dx$$

be the energy associated to the system (3.1) - (3.5) .
Then

$$E'(t) = -\int_0^L \alpha(x)|u_t|^2\, dx$$

and since $\alpha(x) \geq a > 0$ we have

$$E'(t) \leq -a\int_0^L |u_t|^2\, dx \leq 0,$$

that is, the system is dissipative.
With this $E(t)$ in mind, we introduce the following space

$$X := H_0^1(0,L) \times L^2(0,L) \times H_0^1(0,L) \times L^2(0,L) \tag{3.9}$$

endowed with the norm

$$\|U\|_X := \int_0^L \{\rho_1 |v|^2 + \rho_2 |\varphi|^2 + b|\psi_x|^2 + k|u_x - \psi|^2\}dx$$

for $U = (u,v,\psi,\varphi)^T \in X$.
We remark that X is endowed with the scalar product

$$\begin{aligned} &< U_1, U_2 >_X := \rho_1 < v_1, v_2 > + \rho_2 < \varphi_1, \varphi_2 > \\ &+ b < \psi_{1x}, \psi_{2x} > + k < u_{1x} - \psi_1, u_{2x} - \psi_2 > \end{aligned} \tag{3.10}$$

where

$$U_i = \begin{pmatrix} u_i \\ v_i \\ \psi_i \\ \varphi_i \end{pmatrix} \in X, \quad for \ i = 1,2,$$

and $<\cdot,\cdot>$ denotes the scalar product in $L^2(0,L)$.

Thus, X is a Hilbert space.
Define

$$v := u_t$$
$$\varphi := \psi_t .$$

then system (3.1) –3.(5) can be simplified to the following initial value problem or first order evolution equation on X,

$$(AC)\left|\begin{array}{lll} U_t & = & AU(t) \\ U(0) & = & U_0 = (u_o, u_1, \psi_o, \psi_1)^T \in D(A) \end{array}\right. \tag{3.11}$$

with

$$A := \begin{pmatrix} 0 & I & 0 & 0 \\ \frac{k}{\rho_1}(\cdot)_{xx} & -\frac{\alpha(x)}{\rho_1}I & -\frac{k}{\rho_1}(\cdot)_x & 0 \\ 0 & 0 & 0 & I \\ \frac{k}{\rho_2}(\cdot)_x & 0 & -\frac{k}{\rho_2}I + \frac{b}{\rho_2}(\cdot)_{xx} & 0 \end{pmatrix}, \tag{3.12}$$

$$D(A) := (H^2(0,L) \cap H_0^1(0,L)) \times H_0^1(0,L) \times (H^2(0,L) \cap H_0^1(0,L)) \times H_0^1(0,L) \tag{3.13}$$

and $U = (u, v, \psi, \varphi)^T$.
So we have the following result

Theorem 3.1 *The operator* A *defined on (3.12)-(3.13) generates a* C_0 *semigroup of contractions* $(S(t))_{t\geq 0}$ *in the Hilbert space* X.

Proof.- Clearly $D(A)$ is dense in X. Integrating by parts, we have

$$\begin{aligned}
&< AU, U > = \rho_1 < \frac{k}{\rho_1}\{u_{xx} - \psi_x\} - \frac{\alpha(x)}{\rho_1}v, v > \\
&+ \rho_2 < \frac{b}{\rho_2}\psi_{xx} + \frac{k}{\rho_2}(u_x - \psi), \varphi > \\
&+ b < \varphi_x, \psi_x > \\
&+ k < v_x - \varphi, u_x - \psi > \\
&= < k\{u_{xx} - \psi_x\} - \alpha(x)v, v > \\
&+ < b\psi_{xx} + k(u_x - \psi), \varphi > \\
&+ b < \varphi_x, \psi_x > \\
&+ k < v_x - \varphi, u_x - \psi > \\
&= k < u_{xx} - \psi_x, v > - < \alpha(x)v, v > \\
&+ b < \psi_{xx}, \varphi > + k < u_x - \psi, \varphi > \\
&+ b < \varphi_x, \psi_x > \\
&+ k < v_x, u_x - \psi > - k < \varphi, u_x - \psi > \\
&= -k < u_x - \psi, v_x > - < \alpha(x)v, v > \\
&- b < \psi_x, \varphi_x > + b < \psi_x, \varphi_x > \\
&+ k < v_x, u_x - \psi > \\
&= - < \alpha(x)v, v > \\
&\leq -a < v, v > \leq 0,
\end{aligned}$$

then A is dissipative.

We claim that $0 \in \rho(A)$. In fact, we will prove that $\exists A^{-1} \in L(X)$.
Let $F = (f_1, f_2, f_3, f_4) \in X$. We will prove that there is $U \in D(A)$ such that $AU = F$, where $U = (u, v, \psi, \varphi)^T$. Thus, we have

$$v = f_1 \tag{3.14}$$

$$\frac{k}{\rho_1}\{u_{xx} - \psi_x\} - \frac{\alpha(x)}{\rho_1} v = f_2 \tag{3.15}$$

$$\varphi = f_3 \tag{3.16}$$

$$\frac{k}{\rho_2} u_x + \frac{b}{\rho_2}\psi_{xx} - \frac{k}{\rho_2}\psi = f_4. \tag{3.17}$$

Then $v = f_1$ y $\varphi = f_3$. Multiplying equation (3.15) by ρ_1 we get

$$u_{xx} - \psi_x = \frac{\rho_1 f_2 + \alpha(x) f_1}{k}. \tag{3.18}$$

Hence

$$u_x = \psi + H(x). \tag{3.19}$$

Now, multiplying equation (3.17) by ρ_2, we get

$$ku_x + b\psi_{xx} - k\psi = \rho_2 f_4. \tag{3.20}$$

Substituting (3.19) in (3.20), we have

$$\psi_{xx} = \frac{\rho_2 f_4 - kH(x)}{b}. \tag{3.21}$$

Then, by elliptic regularity results $\exists \psi \in H^2 \cap H_0^1$. From (3.18) and elliptic regularity results $\exists u \in H^2 \cap H_0^1$. That is, exists $U \in D(A)$ such that $AU = F$.

Finally, using Theorem 2.1, we conclude that A is the infinitesimal generator of a C_o semigroup of contraction $(S(t))_{t \geq 0}$. And so, the Abstract Cauchy Problem

$$\left| \begin{array}{rcl} U_t & = & AU \\ U(0) & = & U_o \in D(A) \end{array} \right.$$

has one unique solution $U(t) := S(t)U_o$.

4. Exponential Stability

Next theorem is the main result.

Theorem 4.1 *The C_o semigroup of contractions $(S(t))_{t \geq 0}$ generated by A, is exponentially stable.*

Proof.- We will use Theorem 2.2. First, we will prove

$$\rho(A) \supset iR = \{i\beta,\ \beta \in R\}. \tag{4.1}$$

Suppose (4.1) is false, then exists $\beta \in R$ such that $i\beta \in \sigma(A)$.
Since $0 \in \rho(A)$ y A^{-1} is compact, $\sigma(A) = \sigma_p(A)$. That is, the spectral values are eigenvalues.
Then, $i\beta \in \sigma_p(A)$.
Let $U = (u, v, \psi, \varphi)^T \in D(A)$, $U \neq 0$, such that $(i\beta I - A)U = 0$, i.e.

$$AU = i\beta U. \tag{4.2}$$

Using definition of A, (4.2) holds if and only if

$$v = i\beta u \tag{4.3}$$

$$\frac{k}{\rho_1}\{u_{xx} - \psi_x\} - \frac{\alpha(x)}{\rho_1} v = i\beta v \tag{4.4}$$

$$\varphi = i\beta\psi \tag{4.5}$$

$$\frac{k}{\rho_2} u_x + \frac{b}{\rho_2}\psi_{xx} - \frac{k}{\rho_2}\psi = i\beta\varphi, \tag{4.6}$$

that is, $v = i\beta u$ and $\varphi = i\beta\psi$.

Substituting (4.3) in (4.4) and multiplying by ρ_1, we get

$$ku_{xx} - k\psi_x + \rho_1\beta^2 u = i\beta\alpha(x)u. \tag{4.7}$$

Also, substituting (4.5) in (4.6) and multiplying by ρ_2, we have

$$b\psi_{xx} + ku_x - k\psi = -\beta^2\rho_2\psi. \tag{4.8}$$

Now, multiplying equation (4.7) by u, we get

$$ku_{xx}u - k\psi_x u + \rho_1\beta^2 |u|^2 = i\beta\alpha(x)|u|^2$$

and integrating by parts on $[0, L]$, we obtain

$$-k\int_0^L |u_x|^2 + \underbrace{u_x u|_0^L}_{=0} - k\int_0^L \psi_x u + \rho_1\beta^2\int_0^L |u|^2 = i\beta\int_0^L \alpha(x)|u|^2. \tag{4.9}$$

Then

$$\int_0^L \{-k|u_x|^2 - k\psi_x u + \rho_1\beta^2|u|^2\}dx = 0 \tag{4.10}$$

$$\int_0^L \beta\alpha(x)|u|^2\, dx = 0. \tag{4.11}$$

From (4.11), $\beta > 0$ and $\alpha(x) \geq a > 0$, it holds that $\beta\alpha(x)|u|^2 = 0$, which implies $u = 0$. Then, from (4.3) we get $v = 0$, and from (4.10) we arrive to $\int_0^L k|u_x|^2 dx = 0$, i.e. $u_x = 0$. This, from (4.8), leads to

$$b\psi_{xx} - k\psi + \beta^2\rho_2\psi = 0. \tag{4.12}$$

By other hand, from (4.7), we have $-k\psi_x = 0$ and then $\psi_x = 0$ because $k > 0$. Next, by Poincaré inequality, $\psi = 0$ on L^2 . Finally, since ψ is continuous, regular and $\psi(0) = 0$, we have $\psi = 0$. Thus, $u = v = 0 = \psi = \varphi$, which is a contradiction since $U \neq 0$. Therefore, $iR \subset \rho(A)$.

Now, we will prove that

$$\limsup_{|\beta|\to\infty} \left\|(i\beta I - A)^{-1}\right\| < \infty. \tag{4.13}$$

Suppose (4.13) is false, i.e.

$$\limsup_{|\beta|\to\infty} \left\|(i\beta I - A)^{-1}\right\| = \infty, \tag{4.14}$$

then exist sequences $V_n \in X$ and $\beta_n \in R$ such that $\left\|(i\beta_n I - A)^{-1} V_n\right\| \geq n\|\mathrm{V_n}\|$, $\forall n > 0$.

Thus $i\beta_n \in \rho(A)$, or equivalently $\exists (i\beta_n I - A)^{-1} \in L(X)$, that is

$$\exists U_n \in D(A) \text{ such that } (i\beta_n I - A)U_n = V_n, \quad \|U_n\| = 1.$$

So we have

$$U_n = (i\beta_n I - A)^{-1} V_n$$

and

$$\|U_n\| \geq n \Big\| \underbrace{(i\beta_n I - A)U_n}_{G_n :=} \Big\|.$$

Then $1 = \|U_n\| \geq n\|G_n\|$, i.e. $\frac{1}{n} \geq \|G_n\|$. And taking $n \to \infty$, we get $lim_{n \to +\infty} G_n = 0$ on X.

Now, let $U_n := (u_n, v_n, \psi_n, \varphi_n)^T$. Then

$$\begin{aligned} &< G_n, U_n > = < i\beta_n U_n - AU_n, U_n > \\ &= i\beta_n \|U_n\|^2 - < AU_n, U_n >. \end{aligned} \tag{4.15}$$

Taking real part on inequality (4.15), we have

$$-Re < AU_n, U_n > = Re < G_n, U_n >$$

and then

$$\int_0^L \alpha(x) | v_n |^2 \, dx = Re < G_n, U_n > \leq \|G_n\| \underbrace{\|U_n\|}_{=1} = \|G_n\| \to 0 \tag{4.16}$$

since $< AU_n, U_n > = -\int_0^L \alpha(x) | v_n |^2 \, dx$. Thus

$$\int_0^L \alpha(x) | v_n |^2 \, dx \to 0 \text{ when } n \to +\infty. \tag{4.17}$$

Now consider equality (4.15) and multiply by i,

$$-\beta_n \|U_n\|^2 - i \underbrace{< AU_n, U_n >}_{= \int_0^L \alpha(x) |v_n|^2 dx} = i \underbrace{< G_n, U_n >}_{\to 0}. \tag{4.18}$$

Since $|< G_n, U_n >| \leq \|G_n\| \|U_n\| = \|G_n\| \to 0$ and, from (4.17), $\|U_n\|^2 \to 0$ when $n \to +\infty$, we get to $1 = 0$, which is a contradiction.

Thus, by Theorem 2.2, we conclude that $(S(t))_{t \geq 0}$ is exponentially stable.

5. Application: Model with double damping

We will proceed in a similar way for the following system type Timoshenko with double damping

$$\rho_1 u_{tt} - k[u_x - \psi]_x + \alpha(x) u_t = 0, \; (x,t) \in]0, L[\times R^+ \tag{5.1}$$

$$\rho_2 \psi_{tt} - b\psi_{xx} - k[u_x - \psi] + \gamma(x)\psi_t = 0, \; (x,t) \in]0, L[\times R^+ \tag{5.2}$$

$$u(0,t) = u(L,t) = 0, \quad \psi(0,t) = \psi(L,t) = 0, \; t \in R^+ \tag{5.3}$$

$$u(x,0) = u_o(x), \quad u_t(x,0) = u_1(x), \; x \in [0, L] \tag{5.4}$$

$$\psi(x,0) = \psi_o(x), \quad \psi_t(x,0) = \psi_1(x), \; x \in [0, L] \tag{5.5}$$

when $\alpha(x) \geq a > 0$, $\gamma(x) \geq \tilde{a} > 0$.

So, we will prove the existence of solution and exponential stability for this system.

To obtain the energy, multiply equality (5.1) by u_t and integrate on $[0,L]$. Then using integration by parts, we have

$$\frac{1}{2}\frac{\partial}{\partial t}\left\{\int_0^L \rho_1 |u_t|^2 + k|u_x|^2\, dx\right\} - k\int_0^L \psi u_{tx} dx + \int_0^L \alpha(x)|u_t|^2\, dx = 0. \quad (5.6)$$

Also, multiplying (5.2) by ψ_t and integrating on $[0,L]$ we obtain,

$$\frac{1}{2}\frac{\partial}{\partial t}\left\{\int_0^L \rho_2 |\psi_t|^2 + b|\psi_x|^2 + k|\psi|^2\, dx\right\} - k\int_0^L u_x \psi_t dx + \int_0^L \gamma(x)|\psi_t|^2\, dx = 0. \quad (5.7)$$

Summing (5.6) and (5.7) we have

$$\frac{\partial}{\partial t}\underbrace{\frac{1}{2}\left\{\int_0^L \rho_1 |u_t|^2 + \rho_2 |\psi_t|^2 + b|\psi_x|^2 + k|u_x - \psi|^2\, dx\right\}}_{E(t):=} + \int_0^L \alpha(x)|u_t|^2\, dx$$
$$+\int_0^L \gamma(x)|\psi_t|^2\, dx = 0. \quad (5.8)$$

That is

$$\begin{aligned} E'(t) &= -\int_0^L \alpha(x)|u_t|^2\, dx - \int_0^L \gamma(x)|\psi_t|^2\, dx \\ &\le -a\int_0^L |u_t|^2\, dx - \tilde{a}\int_0^L |\psi_t|^2\, dx \\ &\le 0. \end{aligned}$$

That is, our system is dissipative.
Working similarly, we obtain that system (5.1)- (5.5) can be reduced to the following initial value problem or first order evolution equation on X,

$$(AC)\left| \begin{array}{lcl} U_t & = & AU(t) \\ U(0) & = & U_0 = (u_o, u_1, \psi_o, \psi_1)^T \end{array}\right. \quad (5.9)$$

with

$$A := \begin{pmatrix} 0 & I & 0 & 0 \\ \frac{k}{\rho_1}(\cdot)_{xx} & -\frac{\alpha(x)}{\rho_1} I & -\frac{k}{\rho_1}(\cdot)_x & 0 \\ 0 & 0 & 0 & I \\ \frac{k}{\rho_2}(\cdot)_x & 0 & -\frac{k}{\rho_2} I + \frac{b}{\rho_2}(\cdot)_{xx} & -\frac{\gamma(x)}{\rho_2} I \end{pmatrix}, \quad (5.10)$$

$$D(A) := (H^2(0,L) \cap H_0^1(0,L)) \times H_0^1(0,L) \times (H^2(0,L) \cap H_0^1(0,L)) \times H_0^1(0,L) \quad (5.11)$$

and $U = (u, v, \psi, \varphi)^T$.

We have the following result

Theorem 5.1 *Operator* A *defined in (5.10)-(5.11) generates a* C_0 *semigroup of contractions* $(S(t))_{t\ge 0}$ *in the Hilbert space* X.

Proof.- It is similar to the proof of Theorem 3.1. Here

$$\begin{aligned} &< AU, U >= -< \alpha(x)v, v > - < \gamma(x)\varphi, \varphi > \\ &= -a < v, v > -\tilde{a} < \varphi, \varphi > \\ &\le 0 \end{aligned}$$

and equations (3.17) and (3.20) become respectively,

$$\frac{k}{\rho_2}u_x + \frac{b}{\rho_2}\psi_{xx} - \frac{k}{\rho_2}\psi - \frac{\gamma(x)}{\rho_2}\varphi = f_4$$

and

$$ku_x + b\psi_{xx} - k\psi - \gamma(x)\varphi = \rho_2 f_4.$$

Having

$$\psi_{xx} = \frac{\rho_2 f_4 - kH(x) + \gamma(x) f_3}{b}$$

in place of (3.21). Then, using Theorem 2.2 the result follows.

Theorem 5.2 *The* C_o *semigroup of contractions* $(S(t))_{t\geq 0}$ *generated by* A*, is exponentially stable.*

Proof.- Working similarly to the proof of Thorem 4.1, equations (4.6), (4.8) and (4.12), respectively, for this Theorem 5.2 become

$$\frac{k}{\rho_2}u_x + \frac{b}{\rho_2}\psi_{xx} - \frac{k}{\rho_2}\psi - \frac{\gamma(x)}{\rho_2}\varphi = i\beta\varphi,$$

$$b\psi_{xx} + ku_x - k\psi - \gamma(x)(i\beta)\psi = -\beta^2\rho_2\psi,$$

and

$$b\psi_{xx} - k\psi + \beta^2\rho_2\psi - \gamma(x)(i\beta)\psi = 0.$$

By other hand, since $< AU_n, U_n >= -\int_0^L \alpha(x)|v_n|^2\, dx - \int_0^L \gamma(x)|\varphi_n|^2\, dx$, (4.16) becomes

$$\int_0^L \alpha(x)|v_n|^2\, dx + \int_0^L \gamma(x)|\varphi_n|^2\, dx = Re < G_n, U_n > \leq \|G_n\|\|U_n\| = \|G_n\| \to 0.$$

Then, for this Theorem 5.2 (4.17) becomes

$$\int_0^L \alpha(x)|v_n|^2\, dx \to 0 \text{ and}$$

$$\int_0^L \gamma(x)|\varphi_n|^2\, dx \to 0 \text{ when } n \to +\infty. \tag{5.12}$$

Finally, (4.18) for the Theorem 5.2, is

$$-\beta_n\|U_n\|^2 - i\left\{\underbrace{\int_0^L \alpha(x)|v_n|^2\, dx}_{\to 0} + \underbrace{\int_0^L \gamma(x)|\varphi_n|^2\, dx}_{\to 0}\right\} = i\underbrace{< G_n, U_n >}_{\to 0}.$$

Since $|< G_n, U_n >| \leq \|G_n\|\|U_n\| = \|G_n\| \to 0$ and, from (5.12), $\|U_n\|^2 \to 0$ when $n \to +\infty$, we get to $1 = 0$, which is a contradiction.

Thus, by Theorem 2.2, we conclude that $(S(t))_{t\geq 0}$ is exponentially stable.

6. References

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