Estimation and Evaluating of Right Tail Risk

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Abstract

Right tail risk is very important in risk management theory, since it represent the small probability and severity losses event. There are many researches on this topic. This paper focuses on how to estimate and evaluate the right tail risk. The different distributions will be considered. Some numerical examples to illustrate the results will be given.

Key words: Right Tail Risk, Probability Distribution, Mixture Distribution, Losses, ,

1. Introduction

The right tail of a distribution is the part of the distribution corresponding to large values of a random variable. In real life, it means the small probability that have the greatest loss. So it has the great impact on the total loss. Any random variables have the higher probabilities to larger values are said to be heavier-tailed.

One of classification methods of the random variable which only take positive weather to be heavy-tailed distribution or not is determined by the kth raw moment

$$E(X^k) = \int_0^\infty x^k f(x)dx$$

In general agree, if all positive moments exist, the distribution has a light tail, otherwise have heavier tail.

Tail distribution may have great impact on the total loss, therefore many researchers have been done excellent job in this field. See Ahn and Shyamalkumar (2011), Zhu and Li (2012), McNeil, Frey, and Embrechts, (2005), Resnick (2007), and Klugman (2008).

In this paper we are going to discuss risk measurements and tail distributions. We are going to consider mixtures of uniform distribution and also other mixture of the important distribution. In fact, determine the risk measurement of Mixture distribution is really challenging and the numerical evaluation is very hard. Therefore we are going to consider some special cases.

2. Basic Risk Measurements and Background

Many risk measurements have been introduced. Value-at-Risk (VaR) is one of them. VaR used to evaluate exposure to risk, it can be treat as the amount of capital required to ensure with a give degree of certainty, that enterprise doesn't become insolvent. Mathematical definition as follows,

Let X be a loss random variable with a cumulative distribution $F_X(x)$. Then the VaR of a random variable X is 100pth percentile of the distribution of X. Use X_p denote the 100pth percentile of distribution of X, we have

$VaR_p = X_p$ or

$F_X(VaR_p(x)) = F_X(X_p) = p$ or

$P(X > VaR_p(x)) = S_X(X_p) = 1 - p$

Where $S_X(x)$ is the survival function, in order to make sure uniqueness we may use more general definition,

$VaR_p = \inf\{x \in R, F(x) \geq p\}$

Another important risk measurement related to Value-at-Risk which is called Tail-Value-at Risk (TVaR, TCE) or Conditional Tail Expectation (CTE). It is the expected value of loss given that an event outside of a given probability level has occurred.

Mathematical definition as follows,

Let X be a loss random variable, the Tail-Value-at-Risk of X at 100p% level, denoted by $TVaR_p(x)$, and defined by,

$$TVaR_p(x) = E(X \mid X > VaR_p)$$

$$= \frac{\int_{VaR_p}^{\infty} xf(x)dx}{1 - F(VaR_p)}$$

If this quantity is finite, the by integral by parts and change variable, we can easily show that

$$TVaR_p(x) = \frac{\int_p^1 VaR_u(x)du}{1-p}$$

Therefore, Tail-Value-at-Risk can be explained as average of all VaR values above level p. In other words, $TVaR$ tell us information about the tail of the distributions.

We also can rewrite $TVaR$ as follows,

$$TVaR_p(x) = VaR_p + \frac{\int_{VaR_p}^{\infty} (x - VaR_p) f(x)dx}{1-p}$$

Ahn and Shyamalkumar (2011) consider the larger sample behavior of the CTE and VaR, they introduce functions F and F^* satisfies vary complex assumptions, then pointed that for the quantile estimator,

$$\sqrt{n}(Q_\alpha(n, F^*, F) - q_\alpha(F)) \xrightarrow{d} N\left(0, \frac{E_{F^*}(L^2(X)I_{(\xi_\alpha,\infty)}(X)) - (1-\alpha)^2}{f^2(\xi_\alpha)}\right), \text{ as } n \to \infty$$

They also had the result for CTE estimator, which as follows

$$\sqrt{n}(C_\alpha(n, F^*, F) - c_\alpha(F)) \xrightarrow{d} N\left(0, \frac{Var_{F^*}(L(X)I_{(\xi_\alpha,\infty)}(X))(X - \xi_\alpha))}{(1-\alpha)^2}\right), \text{ as } n \to \infty$$

Zhu and Li (2012) considered the Multivariate Tail Conditional Expectation ($TVaR$), and introduced the a strictly increasing homogeneous function with $\psi(cx) = c\psi(x)$ for $c \geq 0$. Then they had the first results

$$E(X_i \mid \psi(\vec{X}) > VaR_p(\psi(\vec{X}))) \propto \frac{\alpha}{\alpha - 1} \frac{E(T_i \psi_+^{\alpha-1}(T))}{E(\psi_+^{\alpha}(T))} VaR_p(\psi(X)), \text{ as } p \to 1,$$

They also gave another result under the certain condition,

$$E(X_i \mid S > t) = \frac{\sigma_{i,S}}{\sigma_S} \frac{\int_{t^2/2\sigma_S^2}^{\infty} g_1(x)dx}{\int_{t/\sigma_S}^{\infty} g_1(x^2/2)dx},$$

3. *Mixture Uniform Distribution and Other Mixture Distributions.*

At first, we consider the simple example. Let $F(x)$ be probability distribution of the equal mixture of the uniform distributions of $U(0,1)$, $U(2,3)$, and $U(4,5)$. Then the probability density function is given as the following

$$f(x) = \begin{cases} 1/3 & 0 \leq x \leq 1 \\ 1/3 & 2 \leq x \leq 3 \\ 1/3 & 4 \leq x \leq 5 \end{cases}$$

It is not hard to find the probability cumulative distribution,

$$F(x) = \begin{cases} 0 & x < 0 \\ x/3 & 0 \le x < 1 \\ 1/3 & 1 \le x < 2 \\ 1/3 + (x-2)/3 & 2 \le x < 3 \\ 2/3 & 3 \le x < 4 \\ 2/3 + (x-4)/3 & 4 \le x < 5 \\ 1 & x \le 5 \end{cases}$$

For $p = 2/3$, then we can easily to find value-at-risk

$$VaR_p(x) = \inf\left\{x \in R, F(x) \ge \frac{2}{3}\right\} = 3$$

and the tail value-at-risk

$$TVaR_p(x) = \frac{\int_p^1 VaR_u(x)du}{1-p}$$

$$= \frac{\int_{2/3}^1 (3u+2)du}{1-\frac{2}{3}} = 4.5$$

Theorem 3.1 Let $x_1 < x_2 < \cdots < x_n$, and $F(x)$ is the probability distribution of the equal mixture of uniform distributions $U(x_1, x_1+1)$, $U(x_2, x_2+1)$, $\cdots, U(x_n, x_n+1)$. For any $p = \frac{J}{n},\ 0 < J < n$. Then the value-at-risk is given by

$VaR_p(x) = x_J + 1$

and the tail value-at-risk is given by

$$TVaR_p(x) = \frac{n}{n-J}\left[\frac{n-J}{2n} + \frac{1}{n}\left(\sum_{k=J+1}^{n} x_k\right)\right]$$

Proof: Since the probability distribution is equal mixture of uniform distributions, so the probability density function is given by

$$f(x) = \frac{1}{n}\sum_{k=1}^{n} I\{[x_k, x_{k+1}]\}$$

where $I\{E\}$ denotes the indicator function of set E. Then we can find $F(x)$ is the piecewise function and nontrivial pieces are given by

$$F(x) = \begin{cases} \frac{J}{n} & x_J + 1 \le x < x_{J+1} \\ \frac{J}{n} + \frac{x - x_{J+1}}{n} & x_{J+1} \le x < x_{J+2} \end{cases},\ J = 1, 2, \cdots, n-2$$

Therefore, for any $p = \frac{J}{n},\ 0 < J < n$.

$$VaR_p(x) = \inf\left\{x \in R, F(x) \ge \frac{2}{3}\right\} = x_J + 1$$

and

$$TVaR_p(x) = \frac{\int_p^1 VaR_u(x)du}{1-p}$$

$$= \frac{1}{1-\frac{J}{n}} \sum_{K=J}^{n-1} \int_{K/n}^{(K+1)/n} (nu - k + x_{k+1})du$$

$$= \frac{n}{n-J}\left[\frac{n-J}{2n} + \frac{1}{n}\left(\sum_{k=J+1}^{n} x_k\right)\right]$$

This finish the proof.

Our simple example is the special case of the theorem 3.1 when $n = 3$, $J = 2$, $x_1 = 0$, $x_2 = 2$, and $x_3 = 4$. According to theorem for $p = 2/3$,

$VaR_p(x) = x_J + 1 = x_2 + 1 = 3$

and

$$TVaR_p(x) = \frac{3}{3-2}\left[\frac{3-2}{6} + \frac{1}{3}(x_3)\right] = 4.5$$

The results exactly same as the simple example results.

Theorem 3.2 Let $x_1 < y_1 < x_2 < y_2 < \cdots < x_n < y_n$, and $F(x)$ is probability distribution of mixture of uniform distributions $U(x_1, y_1)$, $U(x_2, y_2)$, $\cdots, U(x_n, y_n)$. That is,

$f(x) = \sum_{i=1}^{n} k_i \frac{1}{y_i - x_i} I\{[x_i, y_i]\}$, where $\sum_{i=1}^{n} k_i = 1$, and $0 \le k_i \le 1$. For any $p = \sum_{i=1}^{h} k_i$, $0 < h < n$. Then the value-at-risk is given by

$VaR_p(x) = y_h$

and the tail value-at-risk is given by

$$TVaR_p(x) = \frac{1}{2(1-p)} \sum_{i=h+1}^{n} k_i(x_i + y_i)$$

Proof: Based on the probability density function, the cumulative probability distribution $F(x)$ is the piecewise function and nontrivial pieces are given as follows

$$F(x) = \begin{cases} \sum_{i=1}^{h} k_i = p & y_h \le x < x_{h+1} \\ \sum_{i=1}^{h} k_i + k_{h+1}\frac{x - x_{h+1}}{y_{h+1} - x_{h+1}} & x_{h+1} \le x < x_{h+2} \end{cases}, \quad h = 1, 2, \cdots, n-2$$

Therefore, for any $p = \sum_{i=1}^{h} k_i$ $0 < h < n$.

$VaR_p(x) = y_h$

and

$$TVaR_p(x) = \frac{\int_p^1 VaR_u(x)du}{1-p}$$

$$=\frac{1}{1\text{-}\mathrm{p}}\sum_{J=h}^{n-1}\int_{\sum_{i=1}^{J}k_i}^{\sum_{i=1}^{J+1}k_i}\left(\frac{y_{J+1}-x_{J+1}}{k_{J+1}}(u-p)+x_{J+1}\right)du$$

$$=\frac{1}{2(1-p)}\left(\sum_{i=h+1}^{n}k_i\left(x_i+y_i\right)\right)$$

This finish the proof.

One of the special cases of **Theorem 3.2** is, when $k_1=k_2=\cdots=k_n=\frac{1}{n}$, $y_i=x_i+1$, where $i=1,2,\cdots,n$, and $h=J$.

Then $p=\sum_{i=1}^{h}k_i=\frac{J}{n}$

$$TVaR_p(x)=\frac{1}{2(1-\frac{J}{n})}\left(\sum_{i=h+1}^{n}\frac{1}{n}\left(x_i+x_i+1\right)\right)$$

$$=\frac{n}{n-J}\left[\frac{n-J}{2n}+\frac{1}{n}\left(\sum_{k=J+1}^{n}x_k\right)\right]$$

This is the **Theorem 3.2**.

Now we consider another simple example。 Let $F(x)$ be the probability distribution of equal probability mixture of two triangular distribution $T(1;1)$ and $T(3;1)$. The general triangular distribution $T(y;b)$ is given by the following definition

The probability density function is

$$f_{y;b}(x)=\begin{cases}\frac{x-(y-b)}{b^2} & y-b\le x\le y\\ \frac{(y+b)-x}{b^2} & y\le x\le y+b\\ 0 & otherwise\end{cases}$$

Therefore the density function of equal probability mixture of two triangular distribution $T(1;1)$ and $T(3;1)$ is given by

$$f(x)=\begin{cases}x/2 & 0\le x\le 1\\ 1-x/2 & 1\le x\le 2\\ x/2-1 & 2\le x\le 3\\ 2-x/2 & 3\le x\le 4\\ 0 & otherwise\end{cases}$$

and it is not hard to find the cumulative probability distribution is given by

$$F(x)=\begin{cases}0 & x<0\\ x^2/4 & 0\le x\le 1\\ 1/2-(2-x)^2/4 & 1\le x\le 2\\ 1/2+(x-2)^2/4 & 2\le x\le 3\\ 1-(4-x)^2/4 & 3\le x\le 4\\ 1 & x>4\end{cases}$$

Let $p = \frac{1}{2}$, then we can easily to find value-at-risk

$$VaR_p(x) = \inf\left\{x \in R, F(x) \geq \frac{1}{2}\right\} = 2$$

and the tail value-at-risk

$$TVaR_p(x) = \frac{\int_p^1 VaR_u(x)du}{1-p}$$

$$= \frac{\int_{1/2}^{3/4}(2+2(u-0.5)^{0.5})du + \int_{3/4}^{1}(4+2(1-u)^{0.5})du}{1-\frac{1}{2}}$$

$$= 2\left(\frac{2}{3}+\frac{7}{6}\right) = \frac{11}{3}$$

Theorem 3.3 Let $y_1 < y_2 < \cdots < y_n$, and $F(x)$ is the probability distribution of the equal mixture of triangular distribution $T(y_1;1)$, $T(y_2;1)$, $\cdots, T(y_n;1)$. For any $p = \frac{J}{n}$, $0 < J < n$. Then the value-at-risk is given by

$$VaR_p(x) = y_J + 1$$

and the tail value-at-risk is given by

$$TVaR_p(x) = \frac{n}{n-J}\left[\frac{2(n-J)}{3n} + \frac{1}{n}\left(\sum_{k=J+1}^{n} y_k\right)\right]$$

Proof: The triangular probability density function of $T(y_i;1)$ is given by

$$f_{y_i;1}(x) = \begin{cases} x-(y_i-1) & y_i - 1 \leq x \leq y_i \\ (y_i+1)-x & y_i \leq x \leq y_i+1 \\ 0 & otherwise \end{cases}$$

and the corresponding cumulative function is given by

$$F_{y_i;1}(x) = \begin{cases} 0 & x < y_i - 1 \\ \frac{[x-(y_i-1)]^2}{2} & y_i - 1 \leq x \leq y_i \\ 1-\frac{[(y_i+1)-x]^2}{2} & y_i \leq x \leq y_i+1 \\ 1 & x \geq y_i + 1 \end{cases}$$

Therefore the mixture density probability function if $f(x) = \frac{1}{n}\sum_{i=1}^{n} f_{y_i;1}(x)$ and the mixture cumulative function is

$$F(x) = \frac{1}{n}\sum_{i=1}^{n} F_{y_i;1}(x)$$

Therefore, for any $p = \frac{J}{n}$, $0 < J < n$.

$$VaR_p(x) = \inf\left\{x \in R, F(x) \geq \frac{2}{3}\right\} = y_J + 1$$

and

$$TVaR_p(x) = \frac{\int_p^1 VaR_u(x)du}{1-p}$$

$$= \frac{1}{1-\frac{J}{n}}\left\{\sum_{K=J}^{n-1}\int_{K/n}^{K/n+1/2n}\left[y_{k+1}-1+\sqrt{2n}\left(u-\frac{k}{n}\right)^{0.5}\right]du+\int_{K/n+1/2n}^{K/n+1/n}\left[y_{k+1}+1+\sqrt{2n}\left(\frac{k+1}{n}-u\right)^{0.5}\right]du\right\}$$

$$= \frac{n}{n-J}\left\{\sum_{k=J+1}^{n}\left[\frac{(y_k-1)}{2n}+\frac{2}{6n}\right]+\sum_{k=J+1}^{n}\left[\frac{(y_k+1)}{2n}+\frac{2}{6n}\right]\right\}$$

$$= \frac{n}{n-J}\left[\frac{2(n-J)}{3n}+\frac{1}{n}\left(\sum_{k=J+1}^{n}y_k\right)\right]$$

This finish the proof.

Our simple example is the special case of the theorem 3.3 when $n=2$, $J=1$, $y_1=1$, and $y_2=3$. According to theorem 3.3 for $p=1/2$,

$VaR_p(x) = y_J + 1 = y_2 + 1 = 4$

And

$$TVaR_p(x) = \frac{2}{2-1}\left[\frac{2(2-1)}{6}+\frac{1}{2}(y_2)\right] = \frac{11}{3}$$

The results exactly same as the simple example results.

Theorem 3.4 Let $y_1 < y_2 < \cdots < y_n$; $b_i > 0, i = 1,2,\cdots,n;$ $y_i + b_i \le y_{i+1} - b_{i+1}, i = 1,2,\cdots,n-1;$ (this condition guarantees that triangular distribution are not overlap) and $F(x)$ is the probability distribution of the mixture of triangular distribution $T(y_1;b_1)$, $T(y_2;b_2)$, $\cdots, T(y_n;b_n)$. That is $f(x) = \sum_{i=1}^{n} k_i T(y_i, b_i)$, where $\sum_{i=1}^{n} k_i = 1$, and $0 \le k_i \le 1$. For any $p = \sum_{i=1}^{h} k_i$, $0 < h < n$. Then the value-at-risk is given by

$VaR_p(x) = \inf\{x \in R, F(x) \ge p\} = y_h + b_h$

and the tail value-at-risk is given by

$$TVaR_p(x) = \frac{1}{(1-p)}\sum_{J=h+1}^{n} k_J(y_J + \frac{2}{3}b_J)$$

Proof: Based on the probability density function, the cumulative probability distribution $F(x)$ is the piecewise function and nontrivial pieces are given as follows

$$F(x) = \begin{cases} \sum_{i=1}^{h} k_i = p & x \le y_{h+1} - b_{h+1} \\ \sum_{i=1}^{h} k_i + k_{h+1}\frac{(x-(y_{h+1}-b_{h+1}))^2}{2b_i^2} & y_{h+1} - b_{h+1} \le x \le y_{h+1} \ , \ h = 1,2,\cdots,n-2 \\ \sum_{i=1}^{h} k_i + k_{h+1} - k_{h+1}\frac{(y_{h+1}+b_{h+1}-x)^2}{2b_i^2} & y_{h+1} \le x \le y_{h+1} + b_{h+1} \end{cases}$$

Therefore, for any $p=\sum_{i=1}^{h}k_i\ \ 0<h<n$.

$VaR_p(x)=\inf\{x\in R, F(x)\geq p\}=y_h+b_h$

and

$$TVaR_p(x)=\frac{\int_p^1 VaR_u(x)du}{1-p}$$

$$=\frac{1}{1\text{-}\mathrm{p}}\sum_{J=h}^{n-1}\left[\int_{\sum_{i=1}^{J}k_i}^{\sum_{i=1}^{J}k_i+\frac{k_{J+1}}{2}}(y_{J+1}-b_{J+1}+\frac{\sqrt{2}b_{J+1}}{\sqrt{k_{J+1}}}(u-\sum_{i=1}^{J}k_i)^{0.5}du+\int_{\sum_{i=1}^{J}k_i+\frac{k_{J+1}}{2}}^{\sum_{i=1}^{J+1}k_i}(y_{J+1}+b_{J+1}-\frac{\sqrt{2}b_{J+1}}{\sqrt{k_{J+1}}}(\sum_{i=1}^{J+1}k_i-u)^{0.5}du\right]$$

$$=\frac{1}{1-p}\left\{\sum_{J=h+1}^{n}\left[\frac{(y_J-b_J)k_J}{2}+\frac{b_Jk_J}{3}\right]+\sum_{J=h+1}^{n}\left[\frac{(y_J+b_J)k_J}{2}+\frac{b_Jk_J}{3}\right]\right\}$$

$$=\frac{1}{(1-p)}\sum_{J=h+1}^{n}k_J(y_J+\frac{2}{3}b_J)$$

This finish the proof.

One of the special cases of **Theorem 3.4** is, when $k_1=k_2=\cdots=k_n=\frac{1}{n}$, $b_i=1$, where $i=1,2,\cdots,n$, and $h=J$.

Then $p=\sum_{i=1}^{h}k_i=\frac{J}{n}$

$$TVaR_p(x)=\frac{1}{(1-\frac{J}{n})}\left(\sum_{i=h+1}^{n}\frac{1}{n}\left(y_i+\frac{2}{3}\right)\right)$$

$$=\frac{n}{n-J}\left[\frac{2(n-J)}{3n}+\frac{1}{n}\left(\sum_{k=J+1}^{n}y_k\right)\right]$$

This is the result of **Theorem 3.3**.

Example 1: Now let's consider the Pareto distribution. Consider the equal probability mixture of two Pareto distributions with parameters $\alpha=2,\ \ \theta=1;\alpha=2,\theta=2$ respectively. Then we can easily find the mixture cumulative function

$$F(x)=\begin{cases}0 & x<1\\ \frac{1}{2}-\frac{1}{2x^2} & 1\leq x<2\\ 1-\frac{5}{2x^2} & x\geq 2\end{cases}$$

For any $0<p\leq\frac{3}{8}$

$VaR_p(x)=(1-2p)^{-0.5}$

For any $\frac{3}{8}<p<1$

$VaR_p(x)=\frac{\sqrt{5}}{\sqrt{2}}(1-p)^{-0.5}$

For any $0 < p \le \frac{3}{8}$

$$TVaR_p(x) = \frac{\int_p^1 VaR_u(x)du}{1-p}$$

$$= \frac{\int_p^{3/8} (1-2u)^{-0.5} du + \int_{3/8}^1 \frac{\sqrt{5}}{\sqrt{2}} (1-u)^{-0.5} du}{1-p}$$

$$= \frac{1}{1-p}\left[2 + \sqrt{1-2p}\right]$$

Example 2: We may also consider more general case of the Pareto distribution. Consider the equal probability mixture of two Pareto distributions with parameters $\alpha > 1, \ \theta = 1; \ \alpha > 1, \theta = 2$ respectively. Then we also can find the mixture cumulative function

$$F(x) = \begin{cases} 0 & x < 1 \\ \frac{1}{2} - \frac{1}{2x^\alpha} & 1 \le x < 2 \\ 1 - \frac{1+2^\alpha}{2x^\alpha} & x \ge 2 \end{cases}$$

For any $0 < p \le \frac{1}{2} - \frac{1}{2^{\alpha+1}}$

$VaR_p(x) = (1-2p)^{-1/\alpha}$

For any $\frac{1}{2} - \frac{1}{2^{\alpha+1}} < p < 1$

$VaR_p(x) = \left(1+2^\alpha\right)^{1/\alpha} (2-2p)^{-1/\alpha}$

For any $0 < p < \frac{1}{2} - \frac{1}{2^{\alpha+1}}$

$$TVaR_p(x) = \frac{\int_p^1 VaR_u(x)du}{1-p}$$

$$= \frac{\int_p^{1/2-1/2^{\alpha+1}} (1-2u)^{-1/\alpha} du + \int_{1/2-1/2^{\alpha+1}}^1 \left(1+2^\alpha\right)^{1/\alpha} (2-2u)^{-1/\alpha} du}{1-p}$$

$$= \frac{1}{1-p}\left(\frac{\alpha}{\alpha-1}\right)\left[1 + \frac{(1-2p)^{-1/\alpha+1}}{2}\right]$$

The first example is the special case of the second example when $\alpha = 2$. We may get the result

$$TVaR_p(x) = \frac{1}{1-p}\left[2 + \sqrt{1-2p}\right]$$

Theorem 3.5 Let $F(x)$ is the probability distribution of the equal mixture of n Pareto distribution with parameters $\alpha > 1,\ \theta_1 = 1;\ \alpha > 1,\ \theta_2 = 2;\ \cdots;\alpha > 1,\ \theta_n = n;$ respectively.

For any $\frac{k}{n} - \frac{1+2^\alpha+\cdots+k^\alpha}{nk^\alpha} < p \le \frac{k}{n} - \frac{1+2^\alpha+\cdots+k^\alpha}{n(k+1)^\alpha}$; $k = 1,2,\cdots,n$, the value-at-risk is given by

$$VaR_p(x) = \left(1+2^\alpha+\cdots+k^\alpha\right)^{1/\alpha}(k-np)^{-1/\alpha}$$

and the tail value-at-risk is given by

$$TVaR_p(x) = \frac{1}{1-p}\left(\frac{\alpha}{n(\alpha-1)}\right)\left[\left(1+2^\alpha+\cdots+k^\alpha\right)^{1/\alpha}(k-np)^{-1/\alpha+1} + \frac{(k+1+n)(n-k)}{2}\right]$$

Proof: Based on the probability density function of the Pareto distribution, the cumulative probability distribution $F(x)$ is the piecewise function and nontrivial pieces are given as follows

$F(x) = \frac{k}{n} - \frac{1+2^\alpha+\cdots+k^\alpha}{nx^\alpha}$, where $k \le x < k+1$ and $k = 1,2,\cdots,n-1;$

The last piece is given by

$F(x) = 1 - \frac{1+2^\alpha+\cdots+n^\alpha}{nx^\alpha}$, where $x \ge n$.

For any $\frac{k}{n} - \frac{1+2^\alpha+\cdots+k^\alpha}{nk^\alpha} < p \le \frac{k}{n} - \frac{1+2^\alpha+\cdots+k^\alpha}{n(k+1)^\alpha}$; $k = 1,2,\cdots,n$, we set up equation

$$F(x) = \frac{k}{n} - \frac{1+2^\alpha+\cdots+k^\alpha}{nx^\alpha} = p$$

Then solve the equation, we may have the value-at-risk is given by

$$VaR_p(x) = \left(1+2^\alpha+\cdots+k^\alpha\right)^{1/\alpha}(k-np)^{-1/\alpha}$$

Since the cumulative probability distribution is piecewise function, therefore the integral is the sum of the integration of those pieces. The first piece and last piece are different from all middle pieces, so

$$\text{The First Piece} = \frac{\int_p^{\frac{k}{n}-\frac{1+2^\alpha+\cdots+k^\alpha}{n(k+1)^\alpha}} (1+2^\alpha+\cdots+k^\alpha)^{1/\alpha}(k-nu)^{-1/\alpha}\,du}{1-p}$$

$$= \frac{1}{1-p}\left(\frac{\alpha}{n(\alpha-1)}\right)\left[\left(1+2^\alpha+\cdots+k^\alpha\right)^{1/\alpha}(k-np)^{-1/\alpha+1} - \frac{1+2^\alpha+\cdots+k^\alpha}{(k+1)^{\alpha-1}}\right]$$

$$\text{The Middle Pieces} = \frac{\int_{\frac{l}{n}-\frac{1+2^\alpha+\cdots+l^\alpha}{nl^\alpha}}^{\frac{l}{n}-\frac{1+2^\alpha+\cdots+l^\alpha}{n(l+1)^\alpha}} (1+2^\alpha+\cdots+l^\alpha)^{1/\alpha}(l-nu)^{-1/\alpha}\,du}{1-p}$$

$$= \frac{1}{1-p}\left(\frac{\alpha}{\alpha-1}\right)\frac{1+2^\alpha+\cdots+l^\alpha}{n}\left[\frac{1}{l^{\alpha-1}} - \frac{1}{(l+1)^{\alpha-1}}\right],\ \text{where } l = k+1,2,\cdots,n-1;$$

$$\text{The Last Pieces } = \frac{\int_{1-\frac{1+2^{\alpha}+\cdots+n^{\alpha}}{n^{\alpha+1}}}^{1} (1+2^{\alpha}+\cdots+n^{\alpha})^{1/\alpha}(n-nu)^{-1/\alpha}\,du}{1-p}$$

$$= \frac{1}{1-p}\left(\frac{\alpha}{\alpha-1}\right)\frac{1+2^{\alpha}+\cdots+n^{\alpha}}{n}\frac{1}{n^{\alpha-1}}$$

Therefore, the tail value at risk can be found by

$$TVaR_p(x) = \frac{\int_p^1 VaR_u(x)du}{1-p}$$

$$= \frac{1}{1-p}\left(\frac{\alpha}{n(\alpha-1)}\right)\left[\left(1+2^{\alpha}+\cdots+k^{\alpha}\right)^{1/\alpha}(k-np)^{-1/\alpha+1} - \frac{1+2^{\alpha}+\cdots+k^{\alpha}}{(k+1)^{\alpha-1}}\right]$$

$$+\sum_{l=k+1}^{n-1}\frac{1}{1-p}\left(\frac{\alpha}{\alpha-1}\right)\frac{1+2^{\alpha}+\cdots+l^{\alpha}}{n}\left[\frac{1}{l^{\alpha-1}}-\frac{1}{(l+1)^{\alpha-1}}\right]$$

$$+\frac{1}{1-p}\left(\frac{\alpha}{\alpha-1}\right)\frac{1+2^{\alpha}+\cdots+n^{\alpha}}{n}\frac{1}{n^{\alpha-1}}$$

$$= \frac{1}{1-p}\left(\frac{\alpha}{n(\alpha-1)}\right)\left[\left(1+2^{\alpha}+\cdots+k^{\alpha}\right)^{1/\alpha}(k-np)^{-1/\alpha+1} + \sum_{l=k+1}^{n} l\right]$$

$$= \frac{1}{1-p}\left(\frac{\alpha}{n(\alpha-1)}\right)\left[\left(1+2^{\alpha}+\cdots+k^{\alpha}\right)^{1/\alpha}(k-np)^{-1/\alpha+1} + \frac{(k+1+n)(n-k)}{2}\right]$$

This proved the theorem.

Example 2 is the special case of the **Theorem 3.5** when $n=2$, $k=1$, therefore for $0<p<\frac{1}{2}-\frac{1}{2^{\alpha+1}}$,

$VaR_p(x) = (1-2p)^{-1/\alpha}$

and

$$TVaR_p(x) = \frac{1}{1-p}\left(\frac{\alpha}{\alpha-1}\right)\left[1+\frac{(1-2p)^{-1/\alpha+1}}{2}\right]$$

The results exactly same as the example 2 results.

Theorem 3.6: Let $F(x)$ is the probability distribution of the equal mixture of n Pareto distribution with parameters $\alpha>1$, $\theta_1>0$; $\alpha>1$, $\theta_2>0$; $\cdots$; $\alpha>1$, $\theta_n>0$; respectively, and $\theta_1<\theta_2<\cdots<\theta_n$, then for any $\frac{k}{n}-\frac{\theta_1^{\alpha}+\theta_2^{\alpha}+\cdots+\theta_k^{\alpha}}{n\theta_k^{\alpha}} < p \le \frac{k}{n}-\frac{\theta_1^{\alpha}+\theta_2^{\alpha}+\cdots+\theta_k^{\alpha}}{n\theta_{k+1}^{\alpha}}$; $k=1,2,\cdots,n$, the value-at-risk is given by

$VaR_p(x) = \left(\theta_1^{\alpha}+\theta_2^{\alpha}+\cdots+\theta_k^{\alpha}\right)^{1/\alpha}(k-np)^{-1/\alpha}$ and the tail value-at-risk is given by

$$TVaR_p(x) = \frac{1}{1-p}\left(\frac{\alpha}{n(\alpha-1)}\right)\left[\left(\theta_1^{\alpha}+\theta_2^{\alpha}+\cdots+\theta_k^{\alpha}\right)^{1/\alpha}(k-np)^{-1/\alpha+1} + \sum_{l=k+1}^{n}\theta_l\right]$$

Proof: Based on the probability density function of the Pareto distribution, the cumulative probability distribution $F(x)$ is the piecewise function and nontrivial pieces are given as follows

$F(x)=\frac{k}{n}-\frac{\theta_1^\alpha+\theta_2^\alpha+\cdots+\theta_k^\alpha}{nx^\alpha}$, where $\theta_k \le x<\theta_{k+1}$ and $k=1,2,\cdots,n-1$;

The last piece is given by

$F(x)=1-\frac{\theta_1^\alpha+\theta_2^\alpha+\cdots+\theta_n^\alpha}{nx^\alpha}$, where $x\ge\theta_n$

For any $\frac{k}{n}-\frac{\theta_1^\alpha+\theta_2^\alpha+\cdots+\theta_k^\alpha}{n\theta_k^\alpha}<p\le\frac{k}{n}-\frac{\theta_1^\alpha+\theta_2^\alpha+\cdots+\theta_k^\alpha}{n\theta_{k+1}^\alpha}$; $k=1,2,\cdots,n$, we set up equation

$F(x)=\frac{k}{n}-\frac{\theta_1^\alpha+\theta_2^\alpha+\cdots+\theta_k^\alpha}{nx^\alpha}=p$

Then solve the equation, we may have the value-at-risk is given by

$VaR_p(x)=\left(\theta_1^\alpha+\theta_2^\alpha+\cdots+\theta_k^\alpha\right)^{1/\alpha}(k-np)^{-1/\alpha}$

By the similar method in **Theorem 3.5,** we have the following;

The 1st Piece $=\frac{1}{1-p}\frac{\alpha}{n(\alpha-1)}\left[\left(\theta_1^\alpha+\theta_2^\alpha+\cdots+\theta_k^\alpha\right)^{1/\alpha}(k-np)^{-1/\alpha+1}-\frac{\theta_1^\alpha+\theta_2^\alpha+\cdots+\theta_k^\alpha}{\theta_{k+1}^{\alpha-1}}\right]$

The Middle Pieces $=\frac{1}{1-p}\left(\frac{\alpha}{\alpha-1}\right)\frac{\theta_1^\alpha+\theta_2^\alpha+\cdots+\theta_l^\alpha}{n}\left[\frac{1}{\theta_l^{\alpha-1}}-\frac{1}{\theta_{l+1}^{\alpha-1}}\right]$,

where $l=k+1,2,\cdots,n-1$;

The Last Pieces $=\frac{1}{1-p}\left(\frac{\alpha}{\alpha-1}\right)\frac{\theta_1^\alpha+\theta_2^\alpha+\cdots+\theta_n^\alpha}{n}\frac{1}{\theta_n^{\alpha-1}}$

Therefore, the tail value at risk can be found by

$$
\begin{aligned}
TVaR_p(x)&=\frac{\int_p^1 VaR_u(x)du}{1-p}\\
&=\frac{1}{1-p}\frac{\alpha}{n(\alpha-1)}\left[\left(\theta_1^\alpha+\theta_2^\alpha+\cdots+\theta_k^\alpha\right)^{1/\alpha}(k-np)^{-1/\alpha+1}-\frac{\theta_1^\alpha+\theta_2^\alpha+\cdots+\theta_k^\alpha}{\theta_{k+1}^{\alpha-1}}\right]\\
&\quad+\sum_{l=k+1}^{n-1}\frac{1}{1-p}\left(\frac{\alpha}{\alpha-1}\right)\frac{\theta_1^\alpha+\theta_2^\alpha+\cdots+\theta_l^\alpha}{n}\left[\frac{1}{\theta_l^{\alpha-1}}-\frac{1}{\theta_{l+1}^{\alpha-1}}\right]\\
&\quad+\frac{1}{1-p}\left(\frac{\alpha}{\alpha-1}\right)\frac{\theta_1^\alpha+\theta_2^\alpha+\cdots+\theta_n^\alpha}{n}\frac{1}{\theta_n^{\alpha-1}}\\
&=\frac{1}{1-p}\left(\frac{\alpha}{n(\alpha-1)}\right)\left[\left(\theta_1^\alpha+\theta_2^\alpha+\cdots+\theta_k^\alpha\right)^{1/\alpha}(k-np)^{-1/\alpha+1}+\sum_{l=k+1}^{n}\theta_l\right]
\end{aligned}
$$

This proved the theorem.

When we substitute $\theta_k=k$ for $k=1,2,\cdots,n$, we will get the result of **Theorem 3.5**.

Theorem 3.7: Let $F(x)$ is the probability distribution of the arbitrary mixture of n Pareto distribution with parameters $\alpha>1,\ \theta_1>0;\ \alpha>1,\ \theta_2>0;\ \cdots;\alpha>1,\ \theta_n>0;$ respectively, and $\theta_1<\theta_2<\cdots<\theta_n$.

That is $F(x) = \sum_{i=1}^{n} k_i F_i(x)$, where $\sum_{i=1}^{n} k_i = 1$, and $0 \le k_i \le 1$, and $F_i(x)$ is the cumulative probability distribution of the Pareto with $\alpha > 1, \theta_i > 0$; where $i = 1,2,\cdots,n$. Then for any $\sum_{j=1}^{i} k_j - \frac{k_1\theta_1^\alpha + \cdots + k_i\theta_i^\alpha}{\theta_i^\alpha} < p \le \sum_{j=1}^{i} k_j - \frac{k_1\theta_1^\alpha + \cdots + k_i\theta_i^\alpha}{\theta_{i+1}^\alpha}$; $i = 1,2,\cdots,n-1$, the value-at-risk is given by

$$VaR_p(x) = \left(k_1\theta_1^\alpha + k_2\theta_2^\alpha + \cdots + k_i\theta_i^\alpha\right)^{1/\alpha} \left(\sum_{j=1}^{i} k_j - p\right)^{-1/\alpha}$$

and the tail value-at-risk is given by

$$TVaR_p(x) = \frac{1}{1-p}\frac{\alpha}{\alpha-1}\left[\left(k_1\theta_1^\alpha + k_2\theta_2^\alpha + \cdots + k_i\theta_i^\alpha\right)^{1/\alpha} \left(\sum_{j=1}^{i} k_j - p\right)^{-1/\alpha+1} + \sum_{j=i+1}^{n} k_j\theta_j\right]$$

Proof: By the similar method of **Theorem 3.6**, we can prove the results. The **Theorem 3.6** is the special case of **Theorem 3.7** when $k_1 = k_2 = \cdots = k_n = \frac{1}{n}$ and $i = k$.

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