Vibration Analysis of Exponential Cross-Section Beam Using Galerkin's Method

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Abstract

An approximation method is introduced to analyze the problem of uniform and non-uniform beam. Using the differential equation of motion based on the Euler-Bernoulli thin beam equation. The natural frequencies of the beams can be approximate by the Eigen-value of general form of normal mode characteristic function. The Analytical method gives the approximate solution, but need a lot of effort on calculation. The Finite Element Methods is also convenience and popular. This paper studies the vibration analysis in the uniform and exponential cross section beam using the Galerkin's method. The natural frequencies of the beam are determined and, the analysis results are compared to the result from FEM.

Keywords: FEM, Galerkin's Method, Uniform Beam, Non-uniform Beam

1. Introduction

Vibration is an important factor in most of mechanical designs. Since, it may cause, partially or totally, damage in that mechanism. Experienced design engineer consider the possibility and cause of the vibration in the designed system e.g. engine, machine, building, bridge etc. The excitation might be any forces from environment e.g. wind, earthquake or forces from unbalanced of the machinery and, can be either periodical or random. The design engineer must avoid the resonance, lead to the unexpected damages, in the system which is caused by the equality of two frequencies that are frequency of excitation and natural frequency of the system. The vibration analysis of the system needs to be done in the design process.

Most of the standard engineering practice to analyze beams with uniform or variable properties on the basis of Bernoulli-Euler beam theory. In reality, non-uniform beams are presence and the effect of shear distortion and rotary inertia are considered then Timoshenko beam theory is needed [1-2]. As the very advanced computational ability of the computer, the modal analysis can be easily done using the commercial Finite Element software. The analysis results from this software have very good performance in term of time saving and cost saving. Since, they help the manufactures to reduce the design lead time and reduce the enormous number of prototyping cost. However, the solution from the Finite Element Method is not the only one method has been studies. The static deflection of the non-uniform beam has been studied by several different approximation methods e.g. Eigen-function expansion [3], series approximation [4], finite element method [5], transfer matrix method [6], manual approximation method [7], fourth order differential equation [8]. The FEM also has limitations on real-time application e.g. the vibration control because of its long calculation time.

The vibration analysis using approximate analytical method for uniform beam is presented by several researchers [9-11]. This paper presents the vibration analysis of exponential cross section beam using the analytical method. The Galerkin's approximation method [12] is applied. The results of the approximation method are compared to the results from the FEM.

2. Model Condition and Governing Equation

2.1. Model Condition

The considered beams are a uniform cross-section beam and a non-uniform cross-section beam. The dimensions of uniform cross-section beam are 20 mm in width, 200 mm in length.

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The width of the non-uniform beam is increasing along the 200 mm length of beam and is equal to b(x) = $0.02e^{4x}$. Both are 2 mm thick. Material properties used in the analysis are Al-alloy 6061-0's properties. Al-alloy 6061-0 has Young's Modulus (E) of 6.9×1010 N/m^2 and density (ρ) of 2705 kg/m^3. Fig.1 and Fig.2 illustrate shape of the beams.

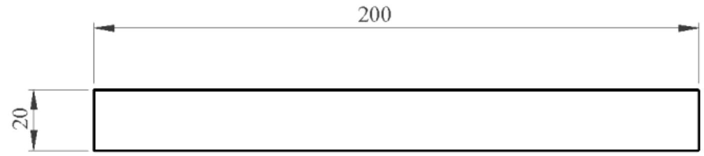

Figure 1 Top view of the uniform beam

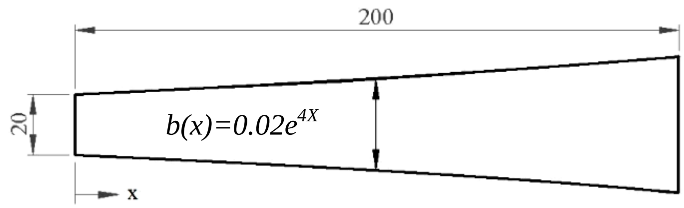

Figure 2 Top view of the non-uniform beam

2.2. Governing Equation

The differential equation of motion for a non-uniform Timoshenko beam subjected to distributed load *f(x, t)* and the static flexural deflection *w(x, t)* can be expressed as (Eq.1).

$$\frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 w}{\partial x^2}(x,t)\right] + \rho A(x)\frac{\partial^2 w}{\partial t^2}(x,t) = f(x,t) \qquad \text{(Eq.1)}$$

Where *E* is Young's modulus, *A(x)* is cross section area of the beam, *I(x)* is the moment of inertia of the beam cross section about y axis and ρ is density. Note that the relationship between bending moment and deflection can be express by (Eq.2), using Euler-Bernoulli thin beam theory.

$$M(x,t) = EI(x)\frac{\partial^2 w}{\partial t^2}(x,t) \qquad \text{(Eq.2)}$$

For the uniform beam (Eq.1) can be reduced to

$$EI\frac{\partial^4 w}{\partial x^4}(x,t) + \rho A\frac{\partial^2 w}{\partial t^2}(x,t) = f(x,t) \qquad \text{(Eq.3)}$$

For the free vibration, *f(x,t) = 0*, the equation of motion is reduced to (Eq.4).

$$c^2\frac{\partial^4 w}{\partial x^4}(x,t) + \frac{\partial^2 w}{\partial t^2}(x,t) = 0 \qquad \text{(Eq.4)}$$

where

$$c = \sqrt{\frac{EI}{\rho A}} \qquad \text{(Eq.5)}$$

The equations of motion with respect to time and distance *x* can be rewritten by the method of separation of variables as shown in (Eq.6).

$$w(x,t) = W(x)\cdot T(t) \qquad \text{(Eq.6)}$$

Substituting (Eq.6) into (Eq.4) and rearrange

$$\frac{c^2}{W(x)}\frac{\partial^4 W(x)}{\partial x^4} = -\frac{1}{T(t)}\frac{\partial^2 T(t)}{\partial t^2} = a' = \omega^2 \qquad \text{(Eq.7)}$$

where $a' = \omega^2$ is a positive constant. (Eq.7) can be rewritten as two equations.

$$\frac{d^4 W(x)}{dx^4} - \beta^4 W(x) = 0 \qquad \text{(Eq.8)}$$

$$\frac{d^2 T(t)}{dt^2} + \omega^2 T(t) = 0 \qquad \text{(Eq.9)}$$

where

$$\beta^4 = \frac{\omega^2}{c^2} = \frac{\rho A \omega^2}{EI} \quad \text{(Eq.10)}$$

where ω is natural frequency of vibration, β is root of the characteristic function. The solution of (Eq.9) can be express as

$$T(t) = A\cos\omega t + B\sin\omega t \quad \text{(Eq.11)}$$

where A and B are constant that can be found from the initial conditions.
The normal mode or characteristic function, *W(x)*, for uniform beam can be solved by assume

$$W(x) = Ce^{sx} \quad \text{(Eq.12)}$$

where C and s are constants, then

$$W(x) = C_1e^{\beta x} + C_2e^{-\beta x} + C_3e^{i\beta x} + C_4e^{-i\beta x} \quad \text{(Eq.13)}$$

and C_1, C_2, C_3, C_4, are constant that can be found from the boundary conditions.
For the non-uniform beam, the (Eq.1) is solved then,

$$W(x) = e^{-2x}(C_1'e^{ax} + C_2'e^{-ax} + C_3'e^{bxi} + C_4'e^{-bxi}) \quad \text{(Eq.14)}$$

where C_1', C_2', C_3', C_4' are constant and

$$a = 0.1\sqrt{400 + 34.3\omega} \quad \text{(Eq.15)}$$

$$b = 0.1\sqrt{-400 + 34.3\omega} \quad \text{(Eq.16)}$$

The natural frequencies of the beam are computed.

$$\omega = (\beta l)^2\sqrt{\frac{EI}{\rho A l^4}} \quad \text{(Eq.17)}$$

2.3 Boundary condition

The models have been subjected to two different boundary conditions, fixed-fixed and fixed-free.

2.3.1 Fixed-Fixed:

Deflection and slope of the beam are equal to zero at both end of the beam.

$$w(0) = 0, \qquad \frac{\partial}{\partial x}w(0) = 0$$

$$w(l) = 0, \qquad \frac{\partial}{\partial x}w(l) = 0$$

2.3.2 Fixed-Free:

Deflection and slope at the fixed end are equal to zero. Bending moment and shear force at the free end is equal to zero.

$$w(0) = 0, \qquad \frac{\partial}{\partial x}w(0) = 0$$

$$EI\frac{\partial^2 w}{\partial x^2} = 0, \qquad \frac{\partial}{\partial x}\left(EI\frac{\partial^2 w}{\partial x^2}\right) = 0$$

3. Solution

3.1 Uniform cross-section beam

The equation of motion (Eq.13) can be rewritten as (Eq.18).

$$w(x) = C_1^* cos\beta x + C_2^* sin\beta x + C_3^* cosh\beta x + C_4^* sinh\beta x \quad \text{(Eq.18)}$$

Substitute the boundary condition of each condition in the equation of motion, the natural frequency of the beam can be solved. (Eq.19) and (Eq.20) represent the solution equations of fixed-fixed condition and fixed-free condition. [14-15]

$$f(\beta l) = cos\beta l \cdot cosh\beta l - 1 \quad \text{(Eq.19)}$$

$$f(\beta l) = cos\beta l \cdot cosh\beta l + 1 \quad \text{(Eq.20)}$$

3.2 Non-uniform cross-section beam

The equation of motion (Eq.14) can be rewrite as shown in (Eq.21).

$$w(x) = e^{-2x}(C_1''\cos ax + C_2''\sin ax + C_3''\cosh bx + C_4''\sinh bx) \quad \text{(Eq.21)}$$

Substitute the boundary condition of each condition in the equation of motion, the natural frequency of the beam can be solved. (Eq.22) and (Eq.23) represent the solution equations of fixed-fixed condition and fixed-free condition.

$$-2A + \left(\frac{A^2}{B} - B\right) sinAlsinhBl + 2AcosAlcoshBl = 0 \quad \text{(Eq.22)}$$

$$[(A^5 - 16A) + (AB^4 - 16A) + (8A^3 + 32A) + (-8AB^2 + 32A) + (24A^2B - \frac{16A^2}{B} + 16AB$$
$$-A^2B^3 + 4B^2 + A^4B + \frac{4A^4}{B})sinAlsinhBl + (2A^3B^2 + 32A + 8AB^2 - 8A^3 - 64A)cosAlcoshBl$$
$$+(\frac{16A^3}{B} + 16AB - 4A^3B)sinhBlcosAl + (-16B^2 - 16A^2 - 4A^2B^2 - 4A^4)\, sinAlcoshBl] = 0 \quad \text{(Eq.23)}$$

4. Result

The natural frequency of the model can be found by solving for the roots of (Eq.8, Eq.9, Eq.11, and Eq.12) and can simply be solved by plot the equation. The roots of the equation can be read form the intersections of x-axis, illustrate by Figure 3(a)-6(a). Figure 3(b)-6(b) illustrate the example of mode shape of beam from FEM result.

Table 1-4 compare the natural frequencies of the beams from approximation method and finite element method.

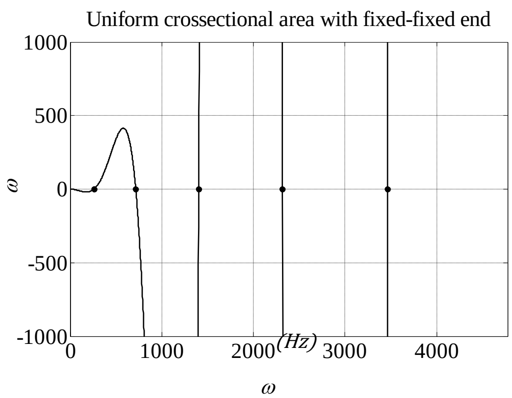

Figure 3(a) Analysis result of uniform beam with fixed-fixed boundary condition

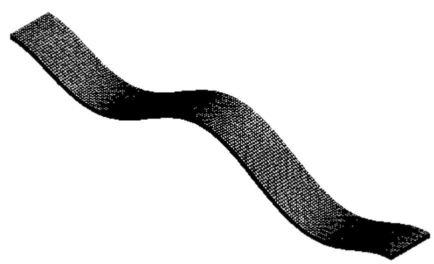

Figure 3(b) Illustrate mode shape of mode 3 of uniform beam with fixed-fixed boundary condition from FEM

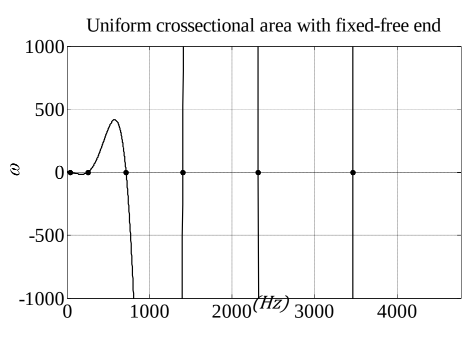

Figure 4(a) Analysis result of uniform beam with fixed-free boundary condition

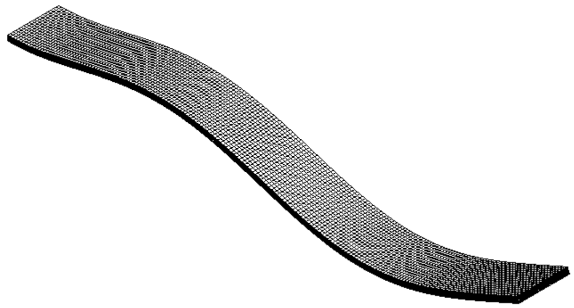

Figure 4(b) Illustrate mode shape of mode 3 of uniform beam with fixed-free boundary condition from FEM

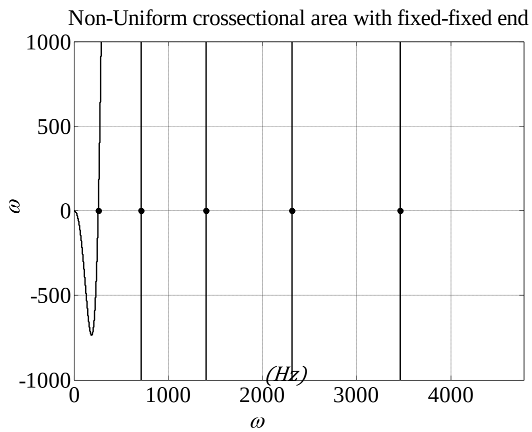

Figure 5(a) Analysis result of non-uniform beam with fixed-fixed boundary condition

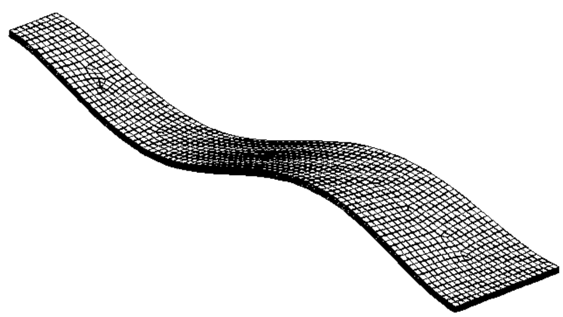

Figure 5(b) Illustrate mode shape of mode 2 of non-uniform beam with fixed-fixed boundary condition from FEM

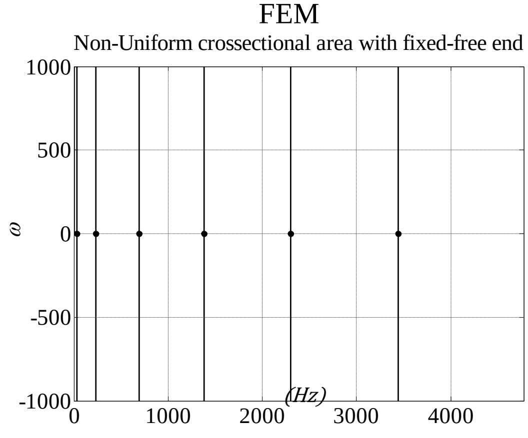

Figure 6(a) Analysis result of non-uniform beam with fixed-free boundary condition

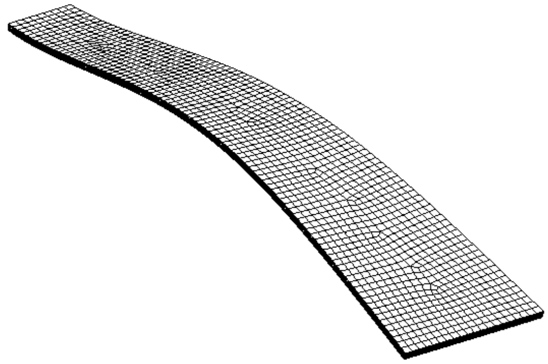

Figure 6(b) Illustrate mode shape of mode 2 of non-uniform beam with fixed-free boundary condition from FEM

Table 1 Natural frequency of the uniform cross-section beam (fixed-fixed)

Mode	ω_n(Hz)		% different
	Analytical	FEM	
1	259.57	263.58	-1.54
2	715.53	725.86	-1.44
3	1402.73	1422.40	-1.40
4	2318.79	2351.10	-1.39
5	3643.83	3512.20	3.61

Table 2 Natural frequency of the uniform cross-section beam (fixed-free)

Mode	ω_n(Hz)		% different
	Analytical	FEM	
1	40.79	41.16	-0.91
2	255.65	257.77	-0.83
3	715.82	721.77	-0.83
4	1402.70	1415.10	-0.88
5	2318.79	2340.70	-0.94

Table 3 Natural frequency of the non-uniform cross-section beam (fixed-fixed)

Mode	ω_n(Hz)		% different
	Analytical	FEM	
1	258.51	266.26	-3.00
2	714.00	731.61	-2.47
3	1400.99	1433.40	-2.31
4	2316.82	2370.70	-2.33
5	3461.78	3543.30	-2.35

Table 4 Natural frequency of the non-uniform cross-section beam (fixed-free)

Mode	ω_n(Hz)		% different
	Analytical	FEM	
1	31.42	32.11	-2.20
2	234.28	239.37	-2.17
3	695.44	706.42	-1.58
4	1382.18	1404.10	-1.59
5	2298.04	2337.10	-1.70

5. Conclusion

This paper presents analytical method based on differential equation of motion for a non-uniform Timoshenko beam and the Euler-Bernoulli thin beam equation.

Two model of continuous beam are consider, the uniform and non-uniform cross-section beam. The boundary conditions used are fixed-fixed and fixed-free. The FEM is used to compare the results. The analytical method and the finite element method give the agreeable results, with slightly different value of the natural frequencies.

Acknowledgement

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