Fully Fuzzy Linear Programming (FFLP) with a Special Ranking Function for Selection of Substitute Activities in Project Management.

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Abstract

An application of Fuzzy Set theory in management sciences is linear programming problems with fuzzy numbers (FLPs). A project manager is always concerned with driving his projects within the time, resource and scope constraints. This study presents a novel selective tool for choosing the activities and the extent to which each activity is executed. We make use of the simplex method to solve (FFLP) problem. A special ranking function used in project environment, is used to rank fuzzy numbers. The constraints that can be used as substitute are considered. Finally those constraint(s) are selected which maximizes the objective function. Project managers can utilize the algorithm for the selection and allocation of resources. The algorithm is effective and reasonable as is evident from the results of a numerical example.

Keywords. Fully Fuzzy Linear Programming Problems (FFLP), Simplex Method, Ranking Function, Planning and Scheduling.

1. Introduction

A max-min operator for converting a fuzzy decision making problem to its crisp equivalent was first proposed by Bellman and Zadeh (1979). The idea was adopted by several authors for solving fuzzy linear programming problems. Rommelfanger *et al.*, (1989), Fang and Hu (1999), Maleki *et al.*, (2000), Maleki (2002) and Khan *et al.*, (2010) are studies where the objective functions, the decision variables, the technical coefficients and the constraints are fuzzy numbers respectively. All these studies considered the partial fuzzy linear programming problems. Nasseri *et al.*, (2005), Ganesan and Veeramani (2006) and Amiri and Nasseri (2007) are some examples utilizing ranking function method for solving linear programming problems without converting the problem to its crisp equivalent.Allahviranloo *et al.*, (2000), Buckley and Feuring (2000), and Hashemi *et al.*, (2006) solved fully fuzzy linear programming problems having inequality constraints. The linear programming problems were converted to its crisp equivalent in all these studies.

A study was conducted by Dehghan *et al.*, (2006) in order to find an exact solution of the fully fuzzy linear programming problems. Lotfi *et al.*, (2009) tried a lexicographic method for solving fully fuzzy linear programming problems and commented that there is no method in the literature to solve a fully fuzzy linear programming problem Mishmast *et al.*, (2004).A new method for solving fully fuzzy linear programming problems with equality constraints by Kumar *et al.*, (2011) using ranking function without converting the problem into its crisp equivalent was proposed by pointing that the final solution by converting the problem to its crisp equivalent is not exact.

This study presents a new method based on the simplex method to find a solution for a fully fuzzy linear programming problem. We make use of a special ranking function that is used in project environment. The uncertainty is modeled with the help of fuzzy triangular numbers. The model is solved directly without converting it into its crisp equivalent along with the use of fuzzy arithmetic.

A fully fuzzy linear programming problem can be written as,

$$\begin{aligned} &\text{Max/Min } \tilde{z} = \tilde{c}^t \tilde{x} \\ &\text{s.t,} \\ &\tilde{A}\tilde{x} \le \tilde{b} \\ &\tilde{x} \ge 0. \end{aligned} \qquad (1)$$

Where $\tilde{z}, c^{\tilde{t}} = (\tilde{c}_1,...\tilde{c}_n), \tilde{x} = (\tilde{x}_1,...\tilde{x}_n), \tilde{A} = [\tilde{a}_{ij}]_{m\times n}, \tilde{b} = (\tilde{b}_1,...\tilde{b}_m)^t$ denote the triangular fuzzy numbers for objective function, fuzzy objective function coefficients, fuzzy technical coefficients and fuzzy resource constraints of the linear programming.

The most important project constraints are Time, Resource and Scope. A deviation of the actual quantities of time, resource and scope from the planned values causes time creep, resource creep and scope creep respectively. A project manager is always concerned with the optimal allocation and utilization of these resources. A weighted averaged estimator for the constrained resources based on three estimates for a resource was proposed in Chu *et al.*, (2005), Vohra (2007) and Waters (2008). This estimator is mathematically given as,

$w_{ij} = \frac{\left[o_{ij} + 4m_{ij} + p_{ij}\right]}{6}$ whereas its variance is $\sigma_{ij} = \frac{\left[p_{ij} - o_{ij}\right]}{6}$.

Here o_{ij} =Optimistic consumption of a resource constraint.

m_{ij} =Most likely consumption of a resource constraint.

p_{ij} =Pessimistic consumption of a resource constraint.

We consider a constraints matrix with triangular fuzzy numbers as,

$$\tilde{A}_{ij} = (o_{ij}, w_{ij}, p_{ij}) \qquad (2)$$

Here o_{ij} and p_{ij} defined above are the optimistic and the pessimistic resource consumption and w_{ij} is the weighted average estimated consumption of the constrained resource.

The variables $\tilde{z}, c^{\tilde{t}} = (\tilde{c}_1,...\tilde{c}_n), \tilde{x} = (\tilde{x}_1,...\tilde{x}_n), \tilde{A} = [\tilde{a}_{ij}]_{m\times n}, \tilde{b} = (\tilde{b}_1,...\tilde{b}_m)^t$ present in equation (1) are triangular fuzzy numbers. Each of which has values as given below.

$\tilde{z} = (z_1, z_2, z_3)$,

$$\begin{array}{lcl} \tilde{c}_1 = (c_{11}, c_{12}, c_{13}) & & \tilde{b}_1 = (b_{11}, b_{12}, b_{13}) \\ \tilde{c}_2 = (c_{21}, c_{22}, c_{23}) & & \tilde{b}_2 = (b_{21}, b_{22}, b_{23}) \\ . & & . \\ . & \text{and} & . \\ . & & . \\ \tilde{c}_n = (c_{n1}, c_{n2}, c_{n3}) & & \tilde{b}_n = (b_{n1}, b_{n2}, b_{n3}) \end{array} \qquad (3)$$

$$\tilde{A}=[\tilde{a}_{ij}]_{m\times n}=\begin{bmatrix} \tilde{a}_{11}=(o_{11},w_{11},p_{11}) & \tilde{a}_{12}=(o_{12},w_{12},p_{12}) & . \quad . \quad . & \tilde{a}_{1n}=(o_{1n},w_{1n},p_{1n}) \\ . & & & \\ . & . & & \\ . & & . \quad . \quad . & \\ \tilde{a}_{m1}=(o_{m1},w_{m1},p_{m1}) & \tilde{a}_{m2}=(o_{m2},w_{m2},p_{m2}) & . \quad . \quad . & \tilde{a}_{mn}=(o_{mn},w_{mn},p_{mn}) \end{bmatrix}$$

The triangular fuzzy numbers represents optimistic, weighted averaged and pessimistic estimates of the parameters.

2. *Arithmetic on Triangular Fuzzy Numbers.*

By Kuafman and Gupta (1988) and Bector and Chandra (2005), a triangular fuzzy number is a triplet T= (*a, b, c*) and is and mathematically,

$$\mu_T(x)=\begin{cases} 0, & x<a \\ \dfrac{x-a}{b-a}, & a\le x<b \\ \dfrac{c-x}{c-b}, & b\le x\le c \\ 0, & x>c \end{cases} \qquad (4)$$

Consider two triangular numbers $\mathrm{T}_1=(a_1,b_1,c_1)$ and $\mathrm{T}_2=(a_2,b_2,c_2)$, the basic arithmetic operations on the triangular numbers are given as under.

1. Image: T= (*a, b, c*): -T= *(-c, -b, -a)*
2. Addition: $\mathrm{T}_1+\mathrm{T}_2=(a_1,b_1,c_1)+(a_2,b_2,c_2)=(a_1+a_2,b_1+b_2,c_1+c_2)$
3. Subtraction: $\mathrm{T}_1+(-\mathrm{T}_2)=(a_1,b_1,c_1)+(-c_2,-b_2,-a_2)=(a_1-c_2,b_1-b_2,c_1-a_2)$
4. Multiplication: *k*.T= (*ka, k b, k c*), k>0.
   k.T= (*kc, k b, k a*), k<0.
   $\mathrm{T}_1.\mathrm{T}_2=(a_1,b_1,c_1)(a_2,b_2,c_2)=(a_1a_2,b_1b_2,c_1c_2)$
5. Division: $\mathrm{T}_1:\mathrm{T}_2=(a_1,b_1,c_1):(\frac{1}{c_2},b_2,\frac{1}{a_2})=(\frac{a_1}{c_2},\frac{b_1}{b_2},\frac{c_1}{a_2})$

3. *Ranking Functions*

A ranking function is used to order fuzzy number and it a mapping from a fuzzy set to the set of real numbers R^n (Naserri *et al.*, 2005). These ranks on R^n are given below.

$$\mathrm{T}_1 \underset{\Re}{\geq} \mathrm{T}_2 \Leftrightarrow \Re(\mathrm{T}_1)\geq\Re(\mathrm{T}_2) \qquad (5)$$

$$\mathrm{T}_1 \underset{\Re}{>} \mathrm{T}_2 \Leftrightarrow \Re(\mathrm{T}_1)>\Re(\mathrm{T}_2) \qquad (6)$$

$$\mathrm{T}_1 \underset{\Re}{=} \mathrm{T}_2 \Leftrightarrow \Re(\mathrm{T}_1)=\Re(\mathrm{T}_2) \qquad (7)$$

$$\text{For } \mathrm{T}_1 \underset{\Re}{\geq} \mathrm{T}_2 \text{ and } \mathrm{T}_3 \underset{\Re}{\geq} \mathrm{T}_4 \text{ then } \Re(\mathrm{T}_1+\mathrm{T}_3)\geq\Re(\mathrm{T}_2+\mathrm{T}_4) \qquad (8)$$

The special ranking function which is used in the study for ordering triangular fuzzy numbers is presented as below.

$$\Re(\mathrm{T})=u+w-\sigma \qquad (9)$$

Where $\sigma=\frac{w-u}{6}$ is the variance between u and w.

4. *The Proposed Simplex Algorithm for Solving Fully Fuzzy Linear Programming Problems.*

Step 1. Given the fully fuzzy linear programming problem as below.

$$\begin{aligned} &\text{Max } \tilde{z} = \tilde{c}^t \tilde{x} \\ &\text{s.t,} \\ &\tilde{A}\tilde{x} \le \tilde{b} \\ &\tilde{x} \ge 0. \end{aligned} \qquad (10)$$

The initial tableau is,

	$\tilde{x}_1$	$\tilde{x}_2$	.	.	.	x_s	.	.	.	$\tilde{x}_n$	$\tilde{x}_{n+1}$	.	.	.	$\tilde{x}_{n+s}$	.	.	.	$\tilde{x}_{n+m}$	$\tilde{b}$
$\tilde{x}_{n+1}$	$\tilde{a}_{12}$	$\tilde{a}_{12}$	.	.	.	$\tilde{a}_{1s}$	.	.	.	$\tilde{a}_{1n}$	1	.	.	.	0	.	.	.	0	$\tilde{b}_1$
$\tilde{x}_{n+2}$	$\tilde{a}_{21}$	$\tilde{a}_{22}$	.	.	.	$\tilde{a}_{2s}$	.	.	.	$\tilde{a}_{2n}$	0	.	.	.	0	.	.	.	0	$\tilde{b}_2$
.	.		.			.	.			.	.	.			.				.	.
.	.			.		.		.		.	.		.		.				.	.
.	.				.	.			.	.	.			.	.				.	.
$\tilde{x}_{n+r}$	$\tilde{a}_{r1}$	$\tilde{a}_{r2}$	.	.	.	$\tilde{a}_{rs}$	.	.	.	$\tilde{a}_{rn}$	0	.	.	.	1				0	$\tilde{b}_r$
.	.	.	.			.	.			.	.	.			.				.	.
.	.	.		.		.		.		.	.		.		.				.	.
.	.	.			.	.			.	.	.			.	.				.	.
$\tilde{x}_{n+m}$	$\tilde{a}_{m1}$	$\tilde{a}_{m2}$	.	.	.	$\tilde{a}_{ms}$	.	.	.	$\tilde{a}_{mn}$	0	.	.	.	0	.	.	.	1	$\tilde{b}_m$
$\tilde{z}$	$-\tilde{c}_1$	$-\tilde{c}_2$	.	.	.	$-\tilde{c}_s$	.	.	.	$-c_n$	0	.	.	.	0	.	.	.	0	

Were1= (1, 1, 1) and 0= (0, 0, 0). $\tilde{x}_{n+i}, i = 1, 2, 3, \ldots m$ are the slack and surplus variables. $\tilde{x}_i, i = 1, 2, 3, \ldots n$ are the non basic decision variables. Initial Basic feasible solution is $[0, 0, \ldots, 0, \ldots, 0, \tilde{b}_1, \tilde{b}_2, \ldots, \tilde{b}_r, \ldots, \tilde{b}_m]^T$.

The augmented matrix of $\tilde{A}\tilde{x} \le \tilde{b}$ is,

$$[\tilde{A} \quad \tilde{I}]\begin{bmatrix} \tilde{x} \\ \tilde{x}_s \end{bmatrix} = \tilde{A}\tilde{x} + \tilde{I}\tilde{x}_s = \tilde{b} \qquad (11)$$

Where $\tilde{x}_s$ represents the vector of slack and surplus variables. The matrix $\tilde{I}$ represents the identity matrix concerning with the columns of the slack and surplus variables. The slack and surplus variables $\{\tilde{x}_{n+i}\}_{i=1}^{m}$ are basic and its initial values are set to be the values of $\tilde{B}$. The matrix $\tilde{N}$ represents the set of non basic variables $\{\tilde{x}_n\}_{i=1}^{m}$.

Step 2. The next step is to replacing $\tilde{x}_r \,\varepsilon\, \tilde{B}$ by $\tilde{x}_s \,\varepsilon\, \tilde{N}$ .The variable $\tilde{x}_r$ leaves the basis and $\tilde{x}_s$ enters the basis so that $\tilde{x}_r$ becomes non basic variable and $\tilde{x}_s$ becomes basic variable. The operation is called pivoting.

Step 3. In case of a maximization problems search for the value of $\tilde{c}_j$ with most negative ranking value. This identifies the pivotal column. Let it be $\tilde{c}_k$.

Step 4. Division of the rank value gained by $\tilde{b}_1, \tilde{b}_2, \ldots, \tilde{b}_r, \ldots, \tilde{b}_m$ by the rank value gained by $\tilde{a}_{1k}, \tilde{a}_{2k}, \ldots, \tilde{a}_{rk}, \ldots, \tilde{a}_{mk}$.

Step 5. Select the least ranking ratio of $\frac{\tilde{b}_1}{\tilde{a}_{1k}}, \frac{\tilde{b}_2}{\tilde{a}_{2k}}, \ldots, \frac{\tilde{b}_r}{\tilde{a}_{rk}}, \ldots, \frac{\tilde{b}_m}{\tilde{a}_{mk}}$ in order to identify the pivotal row. Let it be corresponding to the basic row variable $\tilde{x}_l$.

Step 6. We create a (1, 1, 1) on the position $\tilde{a}_{lk}$ of the pivot column and a (0, 0, 0) above and below it by using Gaussian elimination method along with fuzzy arithmetic. The basic variable $\tilde{x}_l$ leaves the basis and $\tilde{x}_k$ enters the basis.

Step 7. Repeat steps 3-6 until there is no negative ranking value of $\tilde{c}_j$.

5. The Proposed Simplex Method for FLP Problems in Standard Form

A fully fuzzy linear programming problem is given as below.

$$\begin{aligned} &\text{Max } \tilde{z} = \tilde{c}^t \tilde{x} \\ &\text{s.t,} \\ &\tilde{A}\tilde{x} \le \tilde{b} \\ &\tilde{x} \ge 0. \end{aligned} \tag{12}$$

The resource constraints in augmented form is given by,

$$\tilde{A} = [\,\tilde{N}\ \ \tilde{B}]\begin{bmatrix} \tilde{x}_N \\ \tilde{x}_B \end{bmatrix} = \tilde{N}\tilde{x}_N + \tilde{B}\tilde{x}_B = \tilde{b} \tag{13}$$

And the augmented form of the objective function is,

$$\tilde{z} = [\tilde{c}_N^T\ \ \tilde{c}_B^T\,]\begin{bmatrix} \tilde{x}_N \\ \tilde{x}_B \end{bmatrix} = \tilde{c}^T\tilde{x}_N + \tilde{c}_B^T\tilde{x}_B \tag{14}$$

Or,

$$\tilde{z} - \tilde{c}_N^T\tilde{x}_N - \tilde{c}_B^T\tilde{x}_B = 0 \tag{15}$$

Solving (13) for $\tilde{x}_B$

$$\tilde{x}_B = \tilde{B}^{-1}\tilde{b} - \tilde{B}^{-1}\tilde{N}\tilde{x}_N \tag{16}$$

Substituting (16) in (15) and simplifying.

$$\tilde{z} - (\tilde{c}_N^T - \tilde{c}_B^T\tilde{B}^{-1}\tilde{N})\tilde{x}_N = \tilde{c}_B^T\tilde{B}^{-1}\tilde{b} \tag{17}$$

As $\tilde{c}_B^T\tilde{B}^{-1}\tilde{N}$ is a (n-m) row vector, so let us denote it as follows

$$\tilde{z}_N^T = \tilde{c}_B^T\tilde{B}^{-1}\tilde{N} \tag{18}$$

Now (17) becomes,

$$\tilde{z} - (\tilde{c}_N^T - \tilde{z}_N^T)\tilde{x}_N = \tilde{c}_B^T\tilde{B}^{-1}\tilde{b} \tag{19}$$

Now (15) together with (22) gives,

$$\begin{cases} \tilde{B}^{-1}\tilde{N}\tilde{x}_N + \tilde{x}_B = \tilde{B}^{-1}\tilde{b} \\ \tilde{z} - (\tilde{c}_N^T - \tilde{z}_N^T)\tilde{x}_N = \tilde{c}_B^T\tilde{B}^{-1}\tilde{b} \end{cases} \tag{20}$$

Equation (20) generally represents a linear programming problem and can be written in the form,

	$\tilde{x}_N$	$\tilde{x}_B$	$\tilde{b}$
$\tilde{x}_B$	$\tilde{B}^{-1}\tilde{N}$	$\tilde{I}$	$\tilde{B}^{-1}\tilde{b}$
$\tilde{z}$	$-(\tilde{c}_N^T - \tilde{z}_N^T)$	0	$\tilde{c}_B^T\tilde{B}^{-1}\tilde{b}$

In case of a maximization problems the solution is optimal if $(\tilde{c}_j - \tilde{z}_j) \le 0$ and for minimization problems the solution is optimal if $(\tilde{c}_j - \tilde{z}_j) \ge 0$.

6. Dual-Primal Relationship of the Fully Fuzzy Linear Programming Problems.

The primal of a fully fuzzy linear programming problem is,

$$\begin{aligned} &\text{Max } \tilde{z} = \tilde{c}^t \tilde{x} \\ &\text{s.t,} \\ &\tilde{A}\tilde{x} \le \tilde{b} \\ &\tilde{x} \ge 0. \end{aligned} \qquad (21)$$

Its dual can be formulated as,

$$\begin{aligned} &\text{Min } \tilde{z}' = \tilde{b}^t \tilde{y} \\ &\text{s.t,} \\ &\tilde{A}^T \tilde{y} \ge \tilde{c} \\ &\tilde{y} \ge 0. \end{aligned} \qquad (22)$$

From the primal solution one can find the dual solution and vice versa (Kolman and Hill, 2005). The dual solution of a primal can be traced as the coefficients $\tilde{c}_s$ of the $\tilde{z}$ -equation under the columns of the slack variables of the final tableau of the simplex method.

7. Membership Functions for the Objective Function and the Constrained Resources.

The next step is to formulate membership function for the objective function and the constrained resources of the primal and the dual problems. From the solution of equation (21) the membership function of the objective function (z_p, z_w, z_o) can be formulated as,

$$\mu(z(x)) = \begin{cases} \dfrac{z(x) - z_p}{z_w - z_p}, & \text{for } z_p \le z(x) < z_w \\ 1, & \text{for } z(x) = z_w \\ \dfrac{z_o - z(x)}{z_o - z_w}, & \text{for } z_w < z(x) \le z_o \end{cases} \qquad (23)$$

Also the membership function of the constraint is,

$$\mu(x_i) = \begin{cases} \dfrac{x_i - x_p}{x_w - x_p}, & \text{for } x_p \le x_i < x_w \\ 1, & \text{for } x_i = x_w \\ \dfrac{x_o - x_i}{x_o - x_w}, & \text{for } x_w < x_i \le x_o \end{cases} \qquad (24)$$

Similarly for the objective function of the dual problem, we have

$$\mu(z'(x)) = \begin{cases} \dfrac{z'(x) - z'_o}{z'_w - z'_o}, & \text{for } z'_o \le z'(x) < z'_w \\ 1, & \text{for } z'(x) = z'_w \\ \dfrac{z'_p - z'(x)}{z'_p - z'_w}, & \text{for } z'_w < z'(x) \le z'_p \end{cases} \qquad (25)$$

And the membership function for the constraints,

$$\mu(y_i) = \begin{cases} \dfrac{y_i - y_o}{y_w - y_o}, & \text{for } y_o \le y_i < y_w \\ 1, & \text{for } y_i = y_w \\ \dfrac{y_p - y_i}{y_p - y_w}, & \text{for } y_w < y_i \le y_p \end{cases} \qquad (26)$$

Numerical Example:

Let the profit obtained by working on activity "j" of a project is given by the Table1.

Table 1 The Profit obtained by Operating on Activity "j" of a Project.

Activity	(Pess,Mst likly,Opts)	$\tilde{c}_j =$ (Pess,Mst likly,Opts)
1	(2, 5, 8)	(2, 5, 8)
2	(3, 6, 10)	$(3, \frac{37}{6}, 10)$
3	(5,12,15)	$(5, \frac{34}{3}, 15)$

The costs in terms of resources for each of the activity is shown in Table 2. Note that the resources can be used as substitute of each other.

Table 2 The Costs of Operating on each of the Activity of the Project.

Resources	Activity 1 $\tilde{a}_{ij} = (o_{ij}, w_{ij}, p_{ij})$	Activity 2 $\tilde{a}_{ij} = (o_{ij}, w_{ij}, p_{ij})$	Activity 3 $\tilde{a}_{ij} = (o_{ij}, w_{ij}, p_{ij})$	Total Available $\tilde{b}_j =$ (Pess,Mst likly,Opts)
Resource 1	$(2,5,8)$	$(3, \frac{41}{6}, 10)$	$(5, \frac{31}{3}, 18)$	$(6, \frac{50}{3}, 30)$
Resource 2	$(4, \frac{32}{3}, 12)$	$(5, \frac{73}{6}, 20)$	$(7, \frac{105}{6}, 30)$	$(10,30,50)$
Resource 3	$(3,5,7)$	$(5,15,20)$	$(5,10,15)$	$(15, \frac{145}{6}, 30)$

Suppose that $\tilde{x}_j$ represent the amount of activity executed. Then the fully fuzzy linear programming primal problem is formulated as,

$$\text{Max } \tilde{z} = (2,\ 5,\ 8)\tilde{x}_1 + (3, \frac{37}{6}, 10)\tilde{x}_2 + (5, \frac{34}{3}, 15)\tilde{x}_3$$

s.t

$$(2,5,8)\tilde{x}_1 + (3, \frac{41}{6}, 10)\tilde{x}_2 + (5, \frac{31}{3}, 18)\tilde{x}_3 \le (6, \frac{50}{3}, 30)$$

$$(4, \frac{32}{3}, 12)\tilde{x}_1 + (5, \frac{73}{6}, 20)\tilde{x}_2 + (7, \frac{105}{6}, 30)\tilde{x}_3 \le (10,30,50)$$

$$(3,5,7)\tilde{x}_1 + (5,15,20)\tilde{x}_2 + (5,10,15)\tilde{x}_3 \le (15, \frac{145}{6}, 30)$$

$$\tilde{x}_1, \tilde{x}_2, \tilde{x}_3 \ge 0.$$

And its dual is given as,

$$\text{Min } \tilde{z}' = (6, \frac{50}{3}, 30)\tilde{y}_1 + (10, 30, 50)\tilde{y}_2 + (15, \frac{145}{6}, 30)\tilde{y}_3$$

s.t

$$(2,5,8)\tilde{y}_1 + (4, \frac{32}{3}, 12)\tilde{y}_2 + (3,5,7)\tilde{y}_3 \geq (2,\ 5,\ 8)$$

$$(3, \frac{41}{6}, 10)\tilde{y}_1 + (5, \frac{73}{6}, 20)\tilde{y}_2 + (5,15,20)\tilde{y}_3 \geq (3, \frac{37}{6}, 10)$$

$$(5, \frac{31}{3}, 18)\tilde{y}_1 + (7, \frac{105}{6}, 30)\tilde{y}_2 + (5,10,15)\tilde{y}_3 \geq (5, \frac{34}{3}, 15)$$

$$\tilde{y}_1, \tilde{y}_2, \tilde{y}_3 \geq 0.$$

The initial simplex tableau of the primal problem,

	$\tilde{x}_1$	$\tilde{x}_2$	$\tilde{x}_3$	$\tilde{x}_4$	$\tilde{x}_5$	$\tilde{x}_6$	$\tilde{b}$
$\tilde{x}_4\ (R_1)$	$(2,5,8)$	$(3,\frac{41}{6},10)$	$(5,\frac{31}{3},18)$	$(1,1,1)$	$(0,0,0)$	$(0,0,0)$	$(6,\frac{50}{3},30)$
$\tilde{x}_5\ (R_2)$	$(4,\frac{32}{3},12)$	$(5,\frac{73}{6},20)$	$(7,\frac{105}{6},30)$	$(0,0,0)$	$(1,1,1)$	$(0,0,0)$	$(10,30,50)$
$\tilde{x}_6\ (R_3)$	$(3,5,7)$	$(5,15,20)$	$(5,10,15)$	$(0,0,0)$	$(0,0,0)$	$(1,1,1)$	$(15,\frac{145}{6},30)$
$\tilde{z}\ (R_4)$	$(-8,-5,-2)$	$(-10,\frac{-37}{6},-3)$	$(-15,\frac{-34}{3},-5)$	$(0,0,0)$	$(0,0,0)$	$(0,0,0)$	$(0,0,0)$

Step 1: Most negative value of the rank function in the objective function is that of the number $(-15, \frac{-34}{3}, -5)$. So the pivot column is that which corresponding to $\tilde{x}_3$.

Step 2: Divide the rank value gained by the right hand side of the constraints $\tilde{b}_i, i = 1,2,3,...,m$ by the rank value gained by elements of the pivotal column $\tilde{a}_{1k}, \tilde{a}_{2k}, ..., \tilde{a}_{rk}, ..., \tilde{a}_{mk}$ and selecting the least ranking ratio in order to identify the pivot row. The least rank ratio in this case is that of $(5,10,15)$ and $(15, \frac{145}{6}, 30)$. These numbers corresponds to the row corresponding to $\tilde{x}_6$. So $\tilde{x}_3$ enters the basis $\tilde{x}_6$ is the leaves the basis. The number (5, 10, 15) is pivotal element.

Step 3: Now we work for creating a number (1, 1, 1) on the position of pivotal element (5, 10, 15) by the row operation $(\frac{1}{15}, \frac{1}{10}, \frac{1}{5})R_3$ as,

	$\tilde{x}_1$	$\tilde{x}_2$	$\tilde{x}_3$	$\tilde{x}_4$	$\tilde{x}_5$	$\tilde{x}_6$	$\tilde{b}$
$\tilde{x}_4\,(R_1)$	$(2,5,8)$	$(3,\frac{41}{6},10)$	$(5,\frac{31}{3},18)$	$(1,1,1)$	$(0,0,0)$	$(0,0,0)$	$(6,\frac{50}{3},30)$
$\tilde{x}_5\,(R_2)$	$(4,\frac{32}{3},12)$	$(5,\frac{73}{6},20)$	$(7,\frac{105}{6},30)$	$(0,0,0)$	$(1,1,1)$	$(0,0,0)$	$(10,30,50)$
$\tilde{x}_3\,(R_3')$	$(\frac{1}{5},\frac{1}{2},\frac{7}{5})$	$(\frac{1}{3},\frac{17}{12},4)$	$(1,1,1)$	$(0,0,0)$	$(0,0,0)$	$(\frac{1}{15},\frac{1}{10},\frac{1}{5})$	$(1,\frac{29}{12},6)$
$\tilde{z}\,(R_4)$	$(-8,-5,-2)$	$(-10,\frac{-37}{6},-3)$	$(-15,\frac{-34}{3},-5)$	$(0,0,0)$	$(0,0,0)$	$(0,0,0)$	$(0,0,0)$

Step 4: In this step we create number (0, 0, 0) on the positions above and below pivotal element by the operations $R_1+(-18,\frac{-31}{3},-5)R_3'$, $R_2+(-30,\frac{-105}{6},-7)R_3'$ and $R_4+(5,\frac{34}{3},15)R_3'$.

	$\tilde{x}_1$	$\tilde{x}_2$	$\tilde{x}_3$	$\tilde{x}_4$	$\tilde{x}_5$	$\tilde{x}_6$	$\tilde{b}$
$\tilde{x}_4\,(R_1')$	$(\frac{-8}{5},\frac{-1}{6},1)$	$(-3,\frac{-281}{36},-10)$	$(0,0,0)$	$(1,1,1)$	$(0,0,0)$	$(\frac{-6}{5},\frac{-31}{30},-1)$	$(-12,\frac{-299}{36},0)$
$\tilde{x}_5\,(R_2')$	$(-2,\frac{23}{12},\frac{11}{5})$	$(-5,\frac{-101}{8},-8)$	$(0,0,0)$	$(0,0,0)$	$(1,1,1)$	$(-2,\frac{-7}{4},\frac{-7}{5})$	$(-20,\frac{-885}{72},8)$
$\tilde{x}_3\,(R_3')$	$(\frac{1}{5},\frac{1}{2},\frac{7}{5})$	$(\frac{1}{3},\frac{17}{12},4)$	$(1,1,1)$	$(0,0,0)$	$(0,0,0)$	$(\frac{1}{15},\frac{1}{10},\frac{1}{5})$	$(1,\frac{29}{12},6)$
$\tilde{z}\,(R_4')$	$(-7,\frac{2}{3},19)$	$(\frac{-25}{3},\frac{4}{9},57)$	$(0,0,0)$	$(0,0,0)$	$(0,0,0)$	$(\frac{1}{3},\frac{34}{30},3)$	$(5,\frac{986}{36},90)$

This completes the first iteration. Check the optimality condition for the 2nd iteration. It reveals that there is no number in the objective function with negative ranking value, so the solution is optimal and is given below.

$$\tilde{z}=(z_1,z_2,z_3)=(5,\frac{986}{36},90)=(5,27.38,90)$$

at,

$$\tilde{x}_1=(x_{11},x_{12},x_{13})=(0,0,0)$$

$$\tilde{x}_2=(x_{21},x_{22},x_{23})=(0,0,0)$$

$$\tilde{x}_3=(x_{31},x_{32},x_{33})=(1,\frac{29}{12},6)=(1,2.41,6).$$

The coefficients of the objective function under the columns of the slack variables in the final simplex table traced the dual solution and is given by,

$$\tilde{z}'=(z_1',z_2',z_3')=(5,\frac{986}{36},90)=(5,27.38,90)$$

at,

$$\tilde{y}_1=(y_{11},y_{12},y_{13})=(0,0,0)$$

$$\tilde{y}_2=(y_{21},y_{22},y_{23})=(0,0,0)$$

$$\tilde{y}_3=(y_{31},y_{32},y_{33})=(\frac{1}{3},\frac{34}{30},3)=(0.333,1.13,3).$$

The membership functions for the objective function and the constrained resources of the primal and the dual problem are formulated from the results obtained in the last section. In case of the primal problem the membership function of the objective function is as,

$$\mu(z(x))=\begin{cases}\dfrac{z(x)-5}{22.38}, & \text{for } 5\le z(x)<27.38\\ 1, & \text{for } z(x)=27.38\\ \dfrac{90-z(x)}{62.62}, & \text{for } 27.38<z(x)\le 90\end{cases}$$

And that of the constraints resources is,

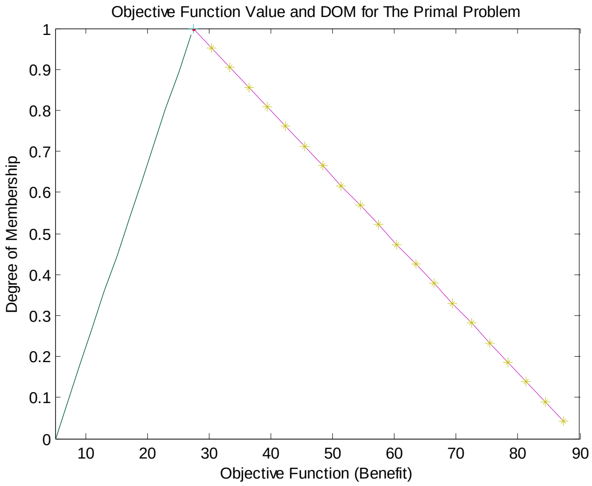
$$\mu(x_3)=\begin{cases}\dfrac{x_3-1}{1.41}, & \text{for } 1\le x_3<2.41\\ 1, & \text{for } x_3=2.41\\ \dfrac{6-x_3}{3.59}, & \text{for } 2.41<x_3\le 6\end{cases}$$

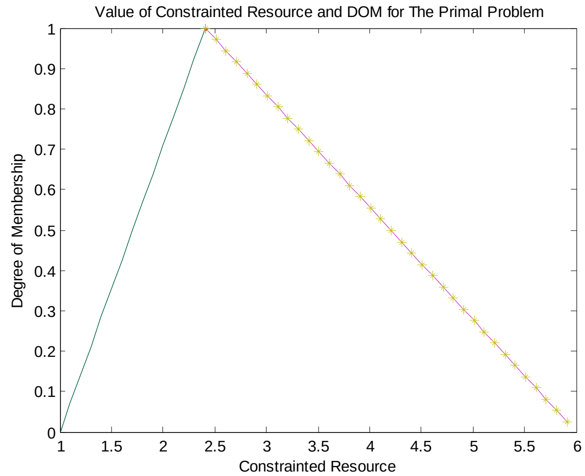
In case of dual problem the membership function for the objective function,

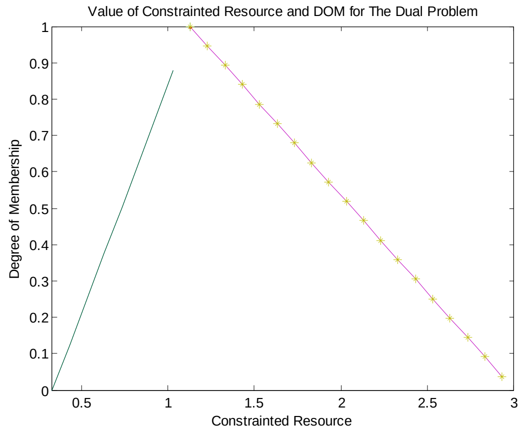
$$\mu(z(x))=\begin{cases}\dfrac{z(x)-5}{22.38}, & \text{for } 5\le z(x)<27.38\\ 1, & \text{for } z(x)=27.38\\ \dfrac{90-z(x)}{62.62}, & \text{for } 27.38<z(x)\le 90\end{cases}$$

Also that of the constraints resources as,

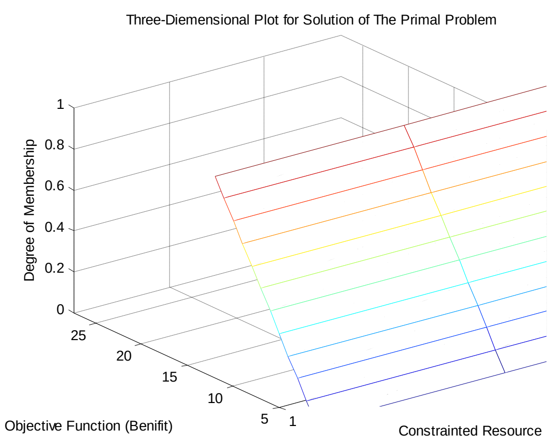
$$\mu(y_3)=\begin{cases}\dfrac{y_3-0.33}{0.13}, & \text{for } 0.33\le y_3<1.13\\ 1, & \text{for } y_3=1.13\\ \dfrac{3-y_3}{1.87}, & \text{for } 1.13<y_3\le 3\end{cases}$$

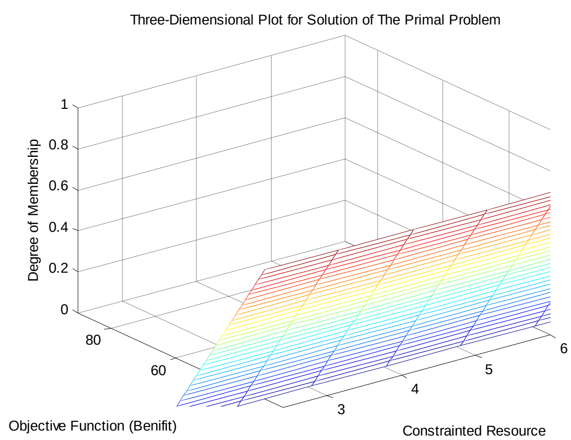


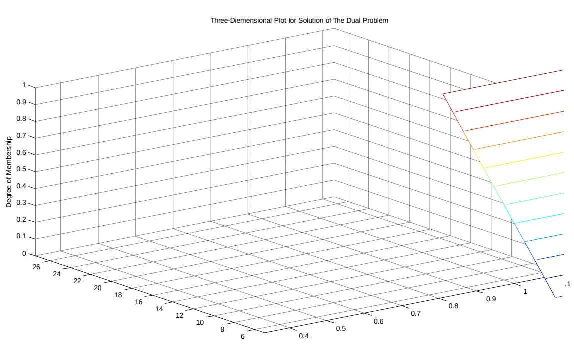


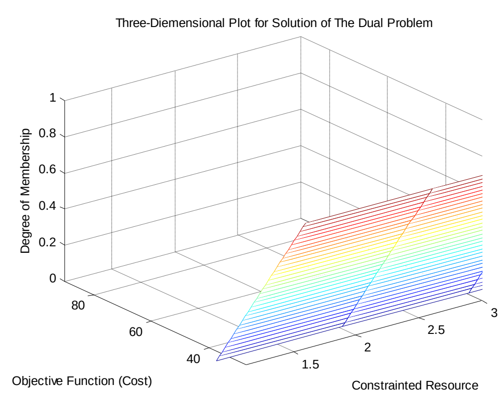


Symbol	Meaning
"-"	Pessimistic Values.
"*"	Optimistic Values.







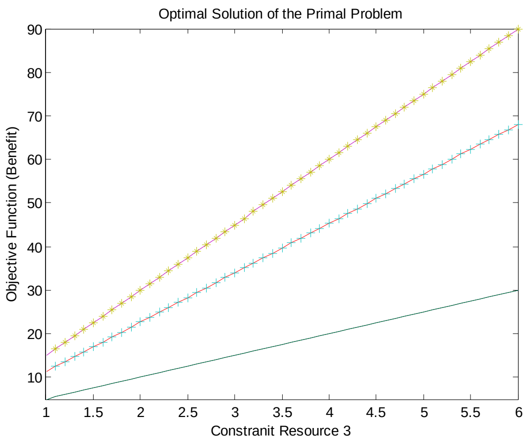


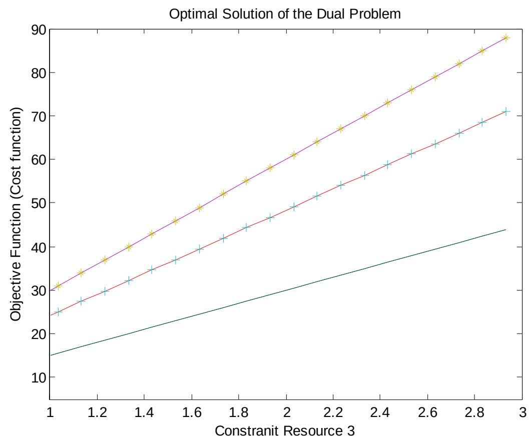
8. Results and Discussion.

The solution of the primal problem reveals that the activity $\tilde{x}_3 = (1, 2.41, 6)$ is executed in the range [1, 6] optimizing the objective function- $\tilde{z} = (5, 27.38, 90)$ in the range of [5, 90]. The solution of the dual problem minimizes the cost of operations on the activities namely $\tilde{y}_1, \tilde{y}_2, \tilde{y}_3$ with a minimized solution in the range of $[5, 90]$ and the activity is executed in the range of $[0.333, 3]$.

Table 3 Primal and Dual solutions of the Fully Fuzzy Linear Programming Problem.

Primal Solution		Dual Solution		$\tilde{z} = (\text{Pess}, \text{Mst likly}, \text{Opts})$
Activity	$(\text{Pess}, \text{Mst likly}, \text{Opts})$	Activity	$(\text{Pess}, \text{Mst likly}, \text{Opts})$	$\tilde{z} = (5, 27.38, 90)$
$\tilde{x}_1$	(0, 0, 0)	$\tilde{y}_1$	(0, 0, 0)	
$\tilde{x}_2$	(0, 0, 0)	$\tilde{y}_2$	(0, 0, 0)	
$\tilde{x}_3$	$(1, 2.41, 6)$	$\tilde{y}_3$	$(0.333, 1.13, 3)$	





Sign Conventions:

	Primal Problem: Symbol	Meaning	Dual Problem: Symbol	Meaning
1.	Line with '-'	Pessimistic value.	Line with '-'	Optimistic Value.
2.	Line with '+'	Weighted Averaged Value.	Line with '+'	Weighted Averaged Value.
3.	Line with '*'	Optimistic Value.	Line with '*'	Pessimistic Value.

The accompanying graphs visualizes the primal and dual solution pointing that the costs and profits of operating on the activity 3 increases as more of the activity is executed. The primal problem maximizes the benefits/profits at $(1, 2.41, 6)$ whereas the dual minimizing the costs of operations at $(0.333, 1.13, 3)$. The triplet alternative solutions (pessimistic, weighted averaged, optimistic) in case of primal and dual solutions are visualized graphically giving an enhanced flexibility, variety and deeper insight to the managers and decision makers.

10. Conclusion

The fully fuzzy linear programming problem is solved using the simplex method. This is a direct method without converting it into its crisp equivalent. Fuzzy numbers are ranked with a special ranking function used in project management. The algorithm can best be applied to the constraints in project management environment that can be used as substitute of each other. Finally the set of substitutes of constraints that optimizes the objective function is selected. The algorithm presents a credible selective and allocative tool for the optimal consumption of the constrained resources.

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