CONTINUOUS DEPENDENCE OF THE SOLUTIONS OF THE DIFFERENTIAL EQUATIONS WITH VARIABLE STRUCTURE AND IMPULSES WITH RESPECT TO THE SWITCHING FUNCTIONS

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Abstract

The initial problem of systems of nonlinear ordinary differential equation with variable structure and impulses is considered in the paper. The changing (switching) in the right-hand side of the system and impulsive effects are realized at the moments, when the switching functions become zero. Sufficient conditions of continuous dependence of the solution on the initial condition and the switching functions are found.

Key words: variable structure, variable impulsive moments, switching functions, continuous dependence

Mathematics Subject Classification: 34A37, 70K20

1. Introduction

The differential equations with variable structure and impulses are convenient mathematical apparatus for modeling the dynamic processes which are subjected to "the intensive and relatively short-term" external influences during its development. It is assumed that the duration of these effects is negligible compared with the total duration of the process, so that it can be considered as "instantaneous", in the form of impulses. Frequently, after these impulsive perturbations, the process continues its development, obeyed to the new rules and laws, different from the previous ones.

Applications of the differential equations with variable structure are mainly in the control theory: [9], [11], [14], [16], [17], [19], [27], [29] and [34]. The impulsive equations are used mostly for describing the species evolution: [6], [8], [13], [14], [15], [21], [22], [23], [24], [26], [30], [31], [32], [33], [35], [37], [38] and [39]. The equations with variable structure and impulses are used to investigate the dynamics of the hydraulic valve stopper in article [10]. The variable structure of the model system corresponds to the both states of the seal valve - "open" and "closed". The impulses are realized at the moments when the seal valve changes its position from “open” to “closed”. In fact, these impulses are realized at the moments when the valve shutter speed is zero, i.e. the seal valve touches its bed.

The moments when, the impulsive effects are materialized and the structure changes can be determined in different ways, which define different classes of the considered systems. We quote the following:

- The switching moments are fixed in advance: [1], [2], [7], [18] and [20];
- The switching moments coincide with the moments at which the integral curve (trajectory) cancels the predefined functions, determined in the phase space of the system differential equations: [5], [14], [21] and [25]. These functions are called switchings;
- The switching moments coincide with the moments, at which the trajectory of the system considered meets the predefined sets, situated in the extended phase space (in general these sets are hypersurfaces): [3], [9], [10], [12], [28] and [32];
- The switching moments coincide with the moments at which the solution minimizes a functional [4];
- The switching moments are random by their nature [36].

In this paper the switching moments are of the second type.

2. Preliminary results

The main object of this paper is the following initial problem of the system of nonlinear ordinary differential equations with variable structure and impulses at non-fixed moments:

(1) $\frac{dx}{dt} = f_i(t,x), \ \varphi_i(x(t)) \neq 0, \ t_{i-1} < t < t_i,$

(2) $\varphi_i(x(t_i)) = 0, \ i = 1,2,...,$

(3) $x(t_i + 0) = x(t_i) + I_i(x(t_i)),$

(4) $x(t_0) = x_0,$

where

- the functions $f_i : R^+ \times D \to R^n$, $f_i = (f_i^1, f_i^2, ... f_i^n)$, the phase space D is non empty domain of R^n;

- the functions $\varphi_i : D \to R$;

- the functions $I_i : D \to R^+$ and $(Id + I_i) : D \to D$, Id is an identity in R^n;

- the initial point $(t_0, x_0) \in R^+ \times D$.

The solution of the initial problem is a partially continuous function, with left continuity at the moments $t_1, t_2, ...$. Moreover, this solution is a diferetiable function in each of the open intervals (t_{i-1}, t_i), $i = 1,2,...$ It is satisfied:

1.1. For t, $t_0 \le t \le t_1$, the solution of the problem (1), (2), (3), (4) coincides with the solution of the initial problem (1), (4) (with invariable structure and without impulses), i.e. coincides with the solution of the problem

(5) $\frac{dx}{dt} = f_1(t,x), \ x(t_0) = x_0;$

1.2. For t, $t_0 < t < t_1$, it is satisfied $\varphi_1(x_1(t)) \neq 0$, where $x_1(t)$ is a solution of the initial problem (5);

1.3. Let t_1 be the first moment after t_0, for which is satisfied the equation $\varphi_1(x_1(t_1)) = 0$;

1.4. At the moment t_1, besides a changing of the right side of the problem considered, the impulsive perturbation of the solution takes place, i.e. it is done the equality (3) for $i = 1$. Ii is valid

$$x(t_1 + 0) = x_1(t_1) + I_1(x_1(t_1)) = (Id + I_1)(x_1(t_1));$$

2.1. For t, $t_1 < t \le t_2$, the solution of the problem (1), (2), (3), (4) coincides with the solution of the initial problem

(6) $\frac{dx}{dt} = f_2(t,x), \ x(t_1 + 0) = (Id + I_1)(x_1(t_1));$

2.2. For t, $t_1 \le t \le t_2$, it is satisfied the inequality $\varphi_2(x_2(t)) \neq 0$, where $x_2(t)$ is a solution of the initial problem (6);

2.3. Let t_2 be the first moment after t_1, for which it is fulfilled the equation $\varphi_2(x_2(t_2)) = 0$;

2.4. At the moment t_2, the right hand side of the problem discussed and the impulsive perturbation takes place, i.e. it is done the equality (3) for $i = 2$. We have

$$x(t_2 + 0) = x_2(t_2) + I_2(x_2(t_2)) = (Id + I_2)(x_2(t_2))$$

etc.

The solution of the problem is left continuous at the moments $t_1, t_2, ...$ In general (for example, when the functions $I_i(x) \neq 0$ for $x \in D$, $i = 1, 2, ...$) this solution has discontinuous right hand side at the points, indicated above. There is a finite jump discontinuity at these points. We will denote, the points $t_1, t_2, ...$ switching moments, the functions I_i, $i = 1, 2, ...$, are impulsive functions and φ_i, $i = 1, 2, ...$, are switching functions.

Further, the solution of the problem (1), (2), (3), (4) we will note with $x(t; t_0, x_0, \varphi_1, \varphi_2, ...)$. More precisely, we have

$$(7)\quad x(t; t_0, x_0, \varphi_1, \varphi_2, ...) = \begin{cases} x(t; t_0, x_0), \ t_0 < t < t_1; \\ x(t; t_0, x_0, \varphi_1), \ t_1 < t < t_2; \\ .... \\ x(t; t_0, x_0, \varphi_1, \varphi_2, ...\varphi_i), \ t_i < t < t_{i+1}; \\ .... \end{cases}$$

With the basic problem we make a study of the so called perturbed initial problem:

$$(8)\quad \frac{dx^*}{dt} = f_i(t, x^*), \quad \varphi_i^*(x^*(t)) \neq 0, \ t_{i-1}^* < t < t_i^*,$$

$$(9)\quad \varphi_i^*(x^*(t_i^*)) = 0, \ i = 1, 2, ...,$$

$$(10)\quad x^*(t_i^* + 0) = x^*(t_i^*) + I_i(x^*(t_i^*)),$$

$$(11)\quad x^*(t_0^*) = x_0^*,$$

where:

- the switching functions $\varphi_i^* : D \to R$;
- the initial point $(t_0^*, x_0^*) \in R^+ \times D$.

The solution of this problem we denote by $x^*(t; t_0^*, x_0^*, \varphi_1^*, \varphi_2^*, ...)$. Like (7) there is

$$x^*(t; t_0^*, x_0^*, \varphi_1^*, \varphi_2^*, ...) = \begin{cases} x^*(t; t_0^*, x_0^*), \ t_0^* < t < t_1^*; \\ x^*(t; t_0^*, x_0^*, \varphi_1^*), \ t_1^* < t < t_2^*; \\ .... \\ x^*(t; t_0^*, x_0^*, \varphi_1^*, \varphi_2^*, ...\varphi_i^*), \ t_i^* < t < t_{i+1}^*; \\ .... \end{cases}$$

Definition 1. *We will say that the solution of the problem (1), (2), (3), (4) depends continuously on the initial condition and the switching functions, if:*

$$(\forall \varepsilon = const > 0)(\forall \eta = const > 0)(\forall T = const > t_0)(\exists \delta = \delta(\varepsilon, \eta, T) > 0):$$

$$(\forall t_0^* \in R^+, \ |t_0^* - t_0| < \delta)(\forall x_0^* \in D, \ \|x_0^* - x_0\| < \delta)$$

$$(\forall \varphi_i^* \in C[D, R], \ \|\varphi_i^*(x) - \varphi_i(x)\| < \delta \ \mathit{for} \ x \in D, \ i = 1, 2, ...) \Rightarrow$$

$$(\|x^*(t; t_0^*, x_0^*, \varphi_1^*, \varphi_2^*, ...) - x(t; t_0, x_0, \varphi_1, \varphi_2, ...)\| < \varepsilon \ \mathit{for} \ t_0^{\max} \leq t \leq T \ \mathit{and} \ |t - t_i| > \eta, \ i = 1, 2, ...),$$

where $t_0^{\max} = \max\{t_0^*, t_0\}$.

Note that the existence of "proximity" between the both solutions (of the problem considered and the corresponding perturbed problem) it not required in the pre-fixed neighbourhoods $(t_i-\eta, t_i+\eta)$, $i=1,2,\ldots$, at the switching moments of the basic problem.

For convenience, we introduce the symbols:

- $\Phi_i=\{x\in D;\ \varphi_i(x)=0\}$, $i=1,2,\ldots$, are switching hypersurfaces of the basic problem;
- $\Phi_i^*=\{x\in D;\ \varphi_i^*(x)=0\}$, $i=1,2,\ldots$, are switching hypersurfaces of the perturbed problem;
- $I_0(x)=0$ for $x\in D$. The equality $(Id+I_0)(x)=x$ is valid;
- $\gamma(t_0,x_0)=\{(t,x(t;t_0,x_0,\varphi_1,\varphi_2,\ldots)),\ t_0\le t\le T\}$ is a trajectory of the problem considered for $t_0\le t\le T$;
- $\gamma^*(t_0^*,x_0^*)=\{(t,x^*(t;t_0^*,x_0^*,\varphi_1^*,\varphi_2^*,\ldots)),\ t_0^*\le t\le T\}$ is a trajectory of the perturbed problem if $t_0\le t\le T$;
- $\|.\|$ and $\langle .,.\rangle$ are the Euclidean norm and the dot product in R^n;
- $B_\delta(x_0)=\{x\in R^n;\ \|x-x_0\|<\delta\}$ is δ - neighbourhood of the point x_0.

We introduce the following conditions:

H1. *The functions* $f_i\in C[R^+\times D, R^n]$ *and there exists the positive constant* C_{Id+I}, *such that for each point* $(t,x)\in R^+\times D$ *and* $i=1,2,\ldots$ *the inequalities*

$$\|f_i(t,x)\|\le C_f$$

are valid.

H2. *The functions* $\varphi_i\in C^1[D,R]$ *and there exists the positive constant* $C_{grad\varphi}$ *such that for each point* $x\in D$ *and* $i=1,2,\ldots$ *the inequalities*

$$\|grad\varphi_i(x)\|\le C_{grad\varphi}$$

are valid.

H3. *The functions* $I_i\in C[D,R^n]$, $(Id+I_i):\Phi_i\to D$ *and there exists the positive constant* $C_{\varphi(Id+I)}$ *such that for each point* $x\in\Phi_i$ *and* $i=1,2,\ldots$ *the inequalities*

$$|\varphi_{i+1}((Id+I_i)(x))|=|\varphi_{i+1}(x+I_i(x))|\ge C_{\varphi(Id+I)}$$

are valid.

H4. *For each point* $(t,x)\in R^+\times D$ *and* $i=1,2,\ldots$ *the inequalities*

$$\varphi_{i+1}((Id+I_i)(x)).\langle grad\varphi_{i+1}(x),\ f_{i+1}(t,x)\rangle<0$$

are valid.

H5. *There exists the positive constant* $C_{\langle grad\varphi, f\rangle}$, *such that for each point* $(t,x)\in R^+\times D$ *and* $i=1,2,\ldots$ *the inequalities*

$$|\langle grad\varphi_i(x),\ f_i(t,x)\rangle|\ge C_{\langle grad\varphi, f\rangle}$$

are valid.

H6. *For each point* $(t_0,x_0)\in R^+\times D$ *and* $i=1,2,\ldots$ *there exists a unique solution of the initial problem for* $t\ge t_0$

(12) $\frac{dx}{dt} = f_i(t,x),\ x(t_0) = x_0$.

Ttheorem 1. *Let the conditions: H1, H2, H3 and H4 hold.*
Then:

1. If the trajectory $\gamma(t_0, x_0)$ *of the problem (1), (2), (3), (4) meets consecutively the switching hypersurfaces* Φ_i *and* Φ_{i+1}*, then for the corresponding switching moments* t_i *and* t_{i+1}*, the following estimate is valid:*

$$t_{i+1} - t_i \geq \frac{C_{\varphi(Id+I)}}{C_{grad\varphi}.C_f}.$$

2. If the trajectory $\gamma(t_0, x_0)$ *meets all the switching hypersurfaces* Φ_i, $i = 1, 2, \ldots$*, then the switching moments increase infinitely, i.e.* $\lim_{i\to\infty} t_i = \infty$.

Proof. Under the conditions H4 the functions $\varphi_{i+1}\left((Id + I_i)(x)\right)$ and $\left\langle grad\varphi_{i+1}(x),\ f_{i+1}(t,x)\right\rangle$ are non-zero and they have opposite signs for any point $(t,x) \in R^+ \times D$. Without loss of generality we assume that the following inequalities are valid:

(13) $\varphi_{i+1}\left((Id + I_i)(x)\right) < 0,\ \ x \in D$

and

(14) $\left\langle grad\varphi_{i+1}(x),\ f_{i+1}(t,x)\right\rangle > 0,\ (t,x) \in R^+ \times D$.

We consider the function $\phi : [t_1, t_{i+1}] \to R$, defined by the equality

$$(15)\qquad \phi(t) = \begin{cases} \varphi_{i+1}\left(x(t_i + 0; t_0, x_0, \varphi_1, \ldots\varphi_i)\right) \\ \qquad = \varphi_{i+1}\left(x(t_i; t_0, x_0, \varphi_1, \ldots\varphi_{i-1}) + I_i\left(x(t_i; t_0, x_0, \varphi_1, \ldots\varphi_{i-1})\right)\right),\ t = t_i; \\ \varphi_{i+1}\left(x(t; t_0, x_0, \varphi_1, \ldots\varphi_i)\right),\ t_i < t \leq t_{i+1}. \end{cases}$$

Under condition H3 and inequality (13) it is true

$$(16)\qquad \phi(t_{i+1}) - \phi(t_i) = \varphi_{i+1}\left(x(t_{i+1}; t_0, x_0, \varphi_1, \ldots\varphi_i)\right) - \varphi_{i+1}\left(x(t_i + 0; t_0, x_0, \varphi_1, \ldots\varphi_i)\right)$$
$$= 0 - \varphi_{i+1}\left(x(t_i; t_0, x_0, \varphi_1, \ldots\varphi_{i-1}) + I_i\left(x(t_i; t_0, x_0, \varphi_1, \ldots\varphi_{i-1})\right)\right)$$
$$= -\varphi_{i+1}\left((Id + I_i)\left(x(t_i; t_0, x_0, \varphi_1, \ldots\varphi_{i-1})\right)\right) = \left|\varphi_{i+1}\left((Id + I_i)\left(x(t_i; t_0, x_0, \varphi_1, \ldots\varphi_{i-1})\right)\right)\right| \geq C_{\varphi(Id+I)}.$$

On the other hand, using (14) and the conditions H1 and H2, we obtain consecutively:

$$\phi(t_{i+1}) - \phi(t_i) = \frac{d}{dt}\phi(\theta)(t_{i+1} - t_i) = \frac{d}{dt}\varphi_{i+1}\left(x(\theta; t_0, x_0, \varphi_1, \ldots\varphi_i)\right).(t_{i+1} - t_i)$$
$$= \left(\frac{\partial}{\partial x_1}\varphi_{i+1}\left(x(\theta; t_0, x_0, \varphi_1, \ldots\varphi_i)\right) f_{i+1}^1\left(\theta, x(\theta; t_0, x_0, \varphi_1, \ldots\varphi_i)\right)\right.$$
$$+ \frac{\partial}{\partial x_2}\varphi_{i+1}\left(x(\theta; t_0, x_0, \varphi_1, \ldots\varphi_i)\right) f_{i+1}^2\left(\theta, x(\theta; t_0, x_0, \varphi_1, \ldots\varphi_i)\right)$$
$$+ \ldots\ldots +$$
$$\left. + \frac{\partial}{\partial x_n}\varphi_{i+1}\left(x(\theta; t_0, x_0, \varphi_1, \ldots\varphi_i)\right) f_{i+1}^n\left(\theta, x(\theta; t_0, x_0, \varphi_1, \ldots\varphi_i)\right)\right).(t_{i+1} - t_i)$$
$$= \left\langle grad\varphi_{i+1}\left(x(\theta; t_0, x_0, \varphi_1, \ldots\varphi_i)\right),\ f_{i+1}\left(\theta, x(\theta; t_0, x_0, \varphi_1, \ldots\varphi_i)\right)\right\rangle.(t_{i+1} - t_i)$$

$$\leq \left\| grad\varphi_{i+1}\left(x\left(\theta;t_0,x_0,\varphi_1,...\varphi_i\right)\right)\right\| . \left\| f_{i+1}\left(\theta,x\left(\theta;t_0,x_0,\varphi_1,...\varphi_i\right)\right)\right\| .\left(t_{i+1}-t_i\right)$$

$$\leq C_{grad\varphi}.C_f.\left(t_{i+1}-t_i\right).$$

The following estimate is achieved from the inequality above

$$t_{i+1}-t_i \geq \frac{1}{C_{grad\varphi}.C_f}\left(\phi\left(t_{i+1}\right)-\phi\left(t_i\right)\right),$$

whence, by means of the inequality (16), it follows that

$$t_{i+1}-t_i \geq \frac{C_{\varphi(Id+I)}}{C_{grad\varphi}.C_f}.$$

If the trajectory of the basic problem meets infinitely many switching hypersurfaces, then using the previous estimate we achieve the conclusion

$$\lim_{i\to\infty} t_i = \lim_{i\to\infty}\left(\left(t_i-t_{i-1}\right)+\left(t_{i-1}-t_{i-2}\right)+...+\left(t_1-t_0\right)+t_0\right) \geq \lim_{i\to\infty} i.\frac{C_{\varphi(Id+I)}}{C_{grad\varphi}.C_f}+t_0=\infty.$$

The theorem is proved.

Theorem 2. *Let the following conditions are fulfilled:*

1. *The conditions: H1, H2, H3, H4 and H5 hold.*
2. *The next inequality is correct for any point* $(t,x)\in R^+\times D$

$$\varphi_1\left(x_0\right).\left\langle grad\varphi_1\left(x\right),\, f_1\left(t,x\right)\right\rangle < 0.$$

Then the trajectory of the problem (1), (2), (3), (4) meets each of the hypersurfaces $\Phi_i,\ i=1,2,...$

Proof. First of all we will show that the trajectory of the basic problem meets the hypersurface Φ_1. One of the following two cases is valid from the condition 2:

Case 1. $\varphi_1\left(x_0\right)<0$, $\left\langle grad\varphi_1\left(x\right),\, f_1\left(t,x\right)\right\rangle>0$ *for* $(t,x)\in R^+\times D$;

Case 2. $\varphi_1\left(x_0\right)>0$, $\left\langle grad\varphi_1\left(x\right),\, f_1\left(t,x\right)\right\rangle<0$ *for* $(t,x)\in R^+\times D$.

Here we will discuss the first case. Another case can be considered in a similar way. We introduce the function $\phi(t)=\varphi_1\left(x\left(t;t_0,x_0\right)\right)$ for $t\geq t_0$. There is

$$\phi\left(t_0\right)=\varphi_1\left(x\left(t_0;t_0,x_0\right)\right)=\varphi_1\left(x_0\right)<0.$$

Under the condition H5 it is satisfied

$$\frac{d}{dt}\phi(t)=\left\langle grad\varphi_1\left(x\left(t;t_0,x_0\right)\right),\ f_1\left(t,x\left(t;t_0,x_0\right)\right)\right\rangle$$

$$=\left|\left\langle grad\varphi_1\left(x\left(t;t_0,x_0\right)\right),\ f_1\left(t,x\left(t;t_0,x_0\right)\right)\right\rangle\right| \geq C_{\langle grad\varphi,f\rangle}=const>0.$$

Using the facts $\phi\left(t_0\right)<0$ and $\frac{d}{dt}\phi(t)=const>0$ for $t>t_0$, it follows that there exists a point $t_1>t_0$, such that $\varphi_1\left(x\left(t_1;t_0,x_0\right)\right)=\phi\left(t_1\right)=0$. This means that at the moment t_1 the trajectory $\gamma\left(t_0,x_0\right)$ meets the hypersurface Φ_1. Assume that the trajectory of the problem considered meets concequtively the hypersurfaces $\Phi_1,\ \Phi_2,...,\Phi_i$ at the moments $t_1,t_2,...,t_i$. Then we prove that $\gamma\left(t_0,x_0\right)$ meets the hypersurface Φ_{i+1}. Once again without loss of generality we assume that the inequalities (13) and (14) are valid. As in the previous theorem, we consider the function ϕ, defined by (15). We have

(17) $\phi(t_i+0)=\varphi_{i+1}\left(x(t_i;t_0,x_0,\varphi_1,...\varphi_{i-1})+I_i\left(x(t_i;t_0,x_0,\varphi_1,...\varphi_{i-1})\right)\right)$

$$=\varphi_{i+1}\left((Id+I_i)\left(x(t_i;t_0,x_0,\varphi_1,...\varphi_{i-1})\right)\right)<0.$$

For $t>t_i$ it is satisfied

(18) $\frac{d}{dt}\phi(t) = \frac{d}{dt}\varphi_{i+1}\left(x(t;t_0,x_0,\varphi_1,...\varphi_i)\right)$

$=\left\langle grad\varphi_{i+1}\left(x(t;t_0,x_0,\varphi_1,...\varphi_i)\right),\ f_{i+1}\left(t,x(t;t_0,x_0,\varphi_1,...\varphi_i)\right)\right\rangle$

$= \left|\left\langle grad\varphi_{i+1}\left(x(t;t_0,x_0,\varphi_1,...\varphi_i)\right),\ f_{i+1}\left(t,x(t;t_0,x_0,\varphi_1,...\varphi_i)\right)\right\rangle\right| \geq C_{\langle grad\varphi,f\rangle}=const>0.$

From (17) and (18) it follows that there exist a point $t_{i+1}>t_i$ such that

$\phi(t_{i+1})=0 \Leftrightarrow \varphi_{i+1}\left(x(t_{i+1};t_0,x_0,\varphi_1,...\varphi_i)\right)=0.$

The interpretation of this equality is that the trajectory of the problem (1), (2), (3), (4) meets the hypersurface Φ_{i+1}. The proof of the theorem follows by induction.

The theorem is proved.

Using theorem 1 and condition H6 we deduce the validity of the next theorem:

Theorem 3. *Let the conditions: H1, H2, H3, H4 and H6 hold.*
Then the solution of the problem (1), (2), (3), (4) exists and it is unique for $t_0 \leq t < \infty$.

Theorem 4. *The following conditions are fulfilled:*
1. The conditions: H1, H2, H3, H4 and H6 hold.
2. For each point $(t,x)\in R^+\times D$ *the following inequality is true*

$\varphi_1(x_0).\left\langle grad\varphi_1(x),\ f_1(t,x)\right\rangle<0.$

3. The trajectory $\gamma(t_0,x_0)$ *of the problem (1), (2), (3), (4) meets the hypersurface* Φ_1 *at the moment* t_1. *Then*

$(\exists\delta=const>0):$

$\left(\forall t_0^*\in R^+,\ |t_0^*-t_0|<\delta\right)\left(\forall x_0^*\in D,\ \|x_0^*-x_0\|<\delta\right)$

$\left(\forall\varphi_1^*\in C[D,R],\ |\varphi_1^*(x)-\varphi_1(x)|<\delta\ for\ x\in D\right)$

$\Rightarrow\gamma^*(t_0^*,x_0^*)\cap\Phi_1^*\neq\varnothing.$

Proof. Under the condition 2 the following inequalities take place:

(19) $\varphi_1(x_0)=\varphi_1\left(x_0+I_0(x_0)\right)=\varphi_1\left((Id+I_0)(x_0)\right)\neq 0,$

$$\left\langle grad\varphi_1(x),\ f_1(t,x)\right\rangle\neq 0,\quad (t,x)\in R^+\times D.$$

Assume that the next inequalities are valid:

(20) $\varphi_1(x_0)=\varphi_1\left(x(t_0;t_0,x_0)\right)<0,\ \varphi_1\left(x(t_1;t_0,x_0)\right)=0$

$$and\ \varphi_1\left(x(t;t_0,x_0)\right)<0\ for\ t_0<t<t_1.$$

The case $\varphi_1(x_0)>0$ is considered in the same way. Assume that for any $t\geq t_0$ there is $\varphi_1\left(x(t;t_0,x_0)\right)\leq 0$. Then t_1 is a point of maximum of the function $\phi(t)=\varphi_1\left(x(t;t_0,x_0)\right)$. It is satisfied

$$0=\frac{d}{dt}\phi(t_1)=\frac{d}{dt}\varphi_1\left(x(t_1;t_0,x_0)\right)=\left\langle grad\varphi_1\left(x(t_1;t_0,x_0)\right),\ f_1\left(t_1,x(t_1;t_0,x_0)\right)\right\rangle.$$

This equality contradicts to the second of the inequalities (19). Therefore there must be a point $\theta>t_1$ such that

$\phi(\theta)=\varphi_1\left(x\left(\theta;t_0,x_0\right)\right)>0$.

From the inequality $\varphi_1\left(x_0\right)<0$ and the continuity of the function φ_1 it follows that there exists a positive constant δ' such that for any point $x\in B_{\delta'}\left(x_0\right)$ the inequality $\varphi_1\left(x\right)<0$ is satisfied. By analogy, using the inequality $\varphi_1\left(x\left(\theta;t_0,x_0\right)\right)>0$ and the continuity of the function φ_1, we obtain the existence of the positive constant δ'' such that for each point $x\in B_{\delta''}\left(x\left(\theta;t_0,x_0\right)\right)$ the inequality $\varphi_1\left(x\right)>0$ is valid. We denote

$$\Delta'=\min\left\{\left|\varphi_1\left(x\right)\right|,\ x\in B_{\delta'}\left(x_0\right)\right\},\ \Delta''=\min\left\{\left|\varphi_1\left(x\right)\right|,\ x\in B_{\delta''}\left(x\left(\theta;t_0,x_0\right)\right)\right\}.$$

In accordance with the classical theorem of continuous dependence of the solutions of the systems differential equations with invariable structure and without impulses on the initial condition (for brevity is called further Classical theorem for continuous dependence) it follows:

$$\left(\exists\delta=const,\ 0<\delta<\delta'\right):\ \left(\forall t_0^*\in R^+,\ \left|t_0^*-t_0\right|<\delta\right)\left(\forall x_0^*\in D,\ \left\|x_0^*-x_0\right\|<\delta\right)$$
$$\Rightarrow\left(\left\|x^*\left(t;t_0^*,x_0^*\right)-x\left(t;t_0,x_0\right)\right\|<\delta''\ \mathit{for}\ t_1^{\max}\le t\le\theta\right).$$

We assume also that for the arbitrary chosen continuous function $\varphi_1^*:D\to R$ is valid in adition the inequality

$$\left|\varphi_1^*\left(x\right)-\varphi_1\left(x\right)\right|<\min\left\{\Delta',\Delta''\right\}\ \mathit{for}\ x\in D\ .$$

For the part of the trajectory $\gamma^*\left(t_0^*,x_0^*\right)$, locked between the points x_0^* and $x^*\left(\theta;t_0^*,x_0^*\right)$, we obtain the following restriction:

1. For the initial point x_0^* it is satisfied

$$\left\|x_0^*-x_0\right\|<\delta\Rightarrow\left\|x_0^*-x_0\right\|<\delta'\Rightarrow x_0^*\in B_{\delta'}\left(x_0\right),$$

which yields

(21) $$\varphi_1^*\left(x_0^*\right)=\varphi_1^*\left(x_0^*\right)-\varphi_1\left(x_0^*\right)+\varphi_1\left(x_0^*\right)$$
$$\le\left|\varphi_1^*\left(x_0^*\right)-\varphi_1\left(x_0^*\right)\right|+\varphi_1\left(x_0^*\right)$$
$$=\left|\varphi_1^*\left(x_0^*\right)-\varphi_1\left(x_0^*\right)\right|-\left|\varphi_1\left(x_0^*\right)\right|<\Delta'-\Delta'=0.$$

2. For the „ final ” point $x^*\left(\theta;t_0^*,x_0^*\right)$ it is valid

$$\left\|x^*\left(t;t_0^*,x_0^*\right)-x\left(t;t_0,x_0\right)\right\|<\delta''\ \mathit{for}\ t_1^{\max}\le t\le\theta$$
$$\Rightarrow\left\|x^*\left(\theta;t_0^*,x_0^*\right)-x\left(\theta;t_0,x_0\right)\right\|<\delta''\Rightarrow x^*\left(\theta;t_0^*,x_0^*\right)\in B_{\delta''}\left(x\left(\theta;t_0,x_0\right)\right).$$

As a result we ascertain that

(22) $$\varphi_1^*\left(x^*\left(\theta;t_0^*,x_0^*\right)\right)=\varphi_1^*\left(x^*\left(\theta;t_0^*,x_0^*\right)\right)-\varphi_1\left(x^*\left(\theta;t_0^*,x_0^*\right)\right)+\varphi_1\left(x^*\left(\theta;t_0^*,x_0^*\right)\right)$$
$$\ge\left|\varphi_1\left(x^*\left(\theta;t_0^*,x_0^*\right)\right)\right|-\left|\varphi_1^*\left(x^*\left(\theta;t_0^*,x_0^*\right)\right)-\varphi_1\left(x^*\left(\theta;t_0^*,x_0^*\right)\right)\right|>\Delta''-\Delta''=0.$$

Under the conditions (21) and (22) for the function $\phi\left(t\right)=\varphi_1^*\left(x^*\left(t;t_0^*,x_0^*\right)\right)$ we find out

$$\phi\left(t_0^*\right)=\varphi_1^*\left(x^*\left(t_0^*;t_0^*,x_0^*\right)\right)=\varphi_1^*\left(x_0^*\right)<0,\quad\phi\left(\theta\right)=\varphi_1^*\left(x^*\left(\theta;t_0^*,x_0^*\right)\right)>0.$$

Using the continuity of the function ϕ we deduce that there exists a point $t_1^*,\ t_0^*<t_1^*<\theta$, such that

$$\phi\left(t_1^*\right)=0 \Leftrightarrow \varphi_1^*\left(x^*\left(t_1^*;t_0^*,x_0^*\right)\right)=0.$$

The meaning of the last equality is that the trajectory $\gamma^*\left(t_0^*,x_0^*\right)$ of the perturbed problem (8), (9), (10), (11) meets the hypersurface Φ_1^* at the moment t_1^*. The theorem is proved.

Theorem 5. *The following conditions are fulfilled:*
1. The conditions: H1, H2, H3, H4 and H6 hold.
2. For each point $(t,x)\in R^+\times D$ the next inequality is valid

$$\varphi_1(x_0).\langle grad\varphi_1(x),\, f_1(t,x)\rangle<0.$$

3. The trajectory $\gamma(t_0,x_0)$ of the problem (1), (2), (3), (4) meets the hypersurface Φ_1 at the moment t_1. Then

$$(\forall\eta=const>0)(\exists\delta=\delta(\eta)>0):$$
$$\left(\forall t_0^*\in R^+,\ \left|t_0^*-t_0\right|<\delta\right)\left(\forall x_0^*\in D,\ \left\|x_0^*-x_0\right\|<\delta\right)$$
$$\left(\forall\varphi_1^*\in C[D,R],\ \left|\varphi_1^*(x)-\varphi_1(x)\right|<\delta \ for\ x\in D\right)$$
$$\Rightarrow\left|t_1^*-t_1\right|<\eta.$$

Proof. Using the Theorem 4 we find out that there exists $\delta'>0$ such that

$$\left(\forall t_0^*\in R^+,\ \left|t_0^*-t_0\right|<\delta'\right)\left(\forall x_0^*\in D,\ \left\|x_0^*-x_0\right\|<\delta'\right)$$
$$\left(\forall\varphi_1^*\in C[D,R],\ \left|\varphi_1^*(x)-\varphi_1(x)\right|<\delta' \ for\ x\in D\right)$$

whence it follows that the trajectory $\gamma^*\left(t_0^*,x_0^*\right)$ of the problem (8), (9), (10), (11) intercepts the perturbed hypersurface Φ_1^* at the moment t_1^*. Under the condition 2 of theorem 5 we assume that the inequalities (20) are valid. Let η be an arbitrary constant and $0<\eta<\min\{t_1-t_0,\, t_2-t_1\}$. Then the following inequalities are valid:

$$\varphi_1\left(x(t;t_0,x_0)\right)<0 \ for\ t_0\le t\le t_1-\eta \quad and \quad \varphi_1\left(x(t_1+\eta;t_0,x_0)\right)>0.$$

We introduce the positive constants:

$$\Delta'=\min\left\{\left|\varphi_1\left(x(t;t_0,x_0)\right)\right|,\, t_0\le t\le t_1-\eta\right\} \ and\ \Delta''=\varphi_1\left(x(t_1+\eta;t_0,x_0)\right).$$

From the first of both inequalities above for $t=t_1-\eta$ it follows that

(23) $\varphi_1\left(x(t_1-\eta;t_0,x_0)\right)>-\Delta'$.

Using the classical theorem of continuous dependence it follows:

(24) $(\exists\delta''=const,\ 0<\delta''<\delta'):\ \left(\forall t_0^*\in R^+,\ \left|t_0^*-t_0\right|<\delta''\right)\left(\forall x_0^*\in D,\ \left\|x_0^*-x_0\right\|<\delta''\right)$

$$\Rightarrow\left\|x^*\left(t;t_0^*,x_0^*\right)-x(t;t_0,x_0)\right\|<\frac{1}{2C_{grad\varphi}}\min\{\Delta',\Delta''\} \ for\ t_1^{\max}\le t\le t_1+\eta.$$

Let the function $\varphi_1^*\in C[D,R]$ and it satisfies

(25) $\left|\varphi_1^*(x)-\varphi_1(x)\right|<\frac{1}{2}\min\{\Delta',\Delta''\} \ for\ x\in D$.

Under condition H2 the gradients of the functions $\varphi_i,\ i=1,2,\ldots$, are bounded above. Hence, the following inequalities are valid:

$$\left|\varphi_i(x')-\varphi_i(x'')\right|\le C_{grad\varphi}\left\|x'-x''\right\|,\quad x',\ x''\in D,\quad i=1,2,\ldots$$

Using the estimates (23), (24) and (25), we obtain:

$$(26) \quad \varphi_1^*\left(x^*\left(t_1-\eta;t_0^*,x_0^*\right)\right)=\varphi_1^*\left(x^*\left(t_1-\eta;t_0^*,x_0^*\right)\right)-\varphi_1\left(x^*\left(t_1-\eta;t_0^*,x_0^*\right)\right)$$

$$+\varphi_1\left(x^*\left(t_1-\eta;t_0^*,x_0^*\right)\right)-\varphi_1\left(x\left(t_1-\eta;t_0,x_0\right)\right)+\varphi_1\left(x\left(t_1-\eta;t_0,x_0\right)\right)$$

$$\le\left|\varphi_1^*\left(x^*\left(t_1-\eta;t_0^*,x_0^*\right)\right)-\varphi_1\left(x^*\left(t_1-\eta;t_0^*,x_0^*\right)\right)\right|$$

$$+C_{grad\varphi}\left\|x^*\left(t_1-\eta;t_0^*,x_0^*\right)-x\left(t_1-\eta;t_0,x_0\right)\right\|+\varphi_1\left(x\left(t_1-\eta;t_0,x_0\right)\right)$$

$$<\frac{1}{2}\Delta'+C_{grad\varphi}\frac{1}{2C_{grad\varphi}}\Delta'-\Delta'=0.$$

On the other hand, we have

$$(27) \quad \varphi_1^*\left(x^*\left(t_1+\eta;t_0^*,x_0^*\right)\right)=\varphi_1^*\left(x^*\left(t_1+\eta;t_0^*,x_0^*\right)\right)-\varphi_1\left(x^*\left(t_1+\eta;t_0^*,x_0^*\right)\right)$$

$$+\varphi_1\left(x^*\left(t_1+\eta;t_0^*,x_0^*\right)\right)-\varphi_1\left(x\left(t_1+\eta;t_0,x_0\right)\right)+\varphi_1\left(x\left(t_1+\eta;t_0,x_0\right)\right)$$

$$>-\left|\varphi_1^*\left(x^*\left(t_1-\eta;t_0^*,x_0^*\right)\right)-\varphi_1\left(x^*\left(t_1-\eta;t_0^*,x_0^*\right)\right)\right|$$

$$-C_{grad\varphi}\left\|x^*\left(t_1-\eta;t_0^*,x_0^*\right)-x\left(t_1-\eta;t_0,x_0\right)\right\|+\varphi_1\left(x\left(t_1-\eta;t_0,x_0\right)\right)$$

$$>-\frac{1}{2}\Delta''-C_{grad\varphi}\frac{1}{2C_{grad\varphi}}\Delta''-\Delta''=0.$$

We rewrite (26) and (27) in a more compact form: $\phi\left(t_1-\eta\right)<0$ and $\phi\left(t_1+\eta\right)>0$ respectively. From the continuity of the function $\phi\left(t\right)=\varphi_1^*\left(x^*\left(t;t_0^*,x_0^*\right)\right)$ for $t_1-\eta\le t\le t_1+\eta$ it follows that there exists a point t_1^*,

$$t_1-\eta<t_1^*<t_1+\eta \Leftrightarrow -\eta<t_1^*-t_1<\eta \Leftrightarrow \left|t_1^*-t_1\right|<\eta,$$

such that

$$\phi\left(t_1^*\right)=\varphi_1^*\left(x^*\left(t_1^*;t_0^*,x_0^*\right)\right)=0.$$

The last equality means that the trajectory $\gamma^*\left(t_0^*,x_0^*\right)$ of the perturbed problem intercepts the hypersurface Φ_1^* at the moment t_1^*, for which it is satisfied the inequality $\left|t_1^*-t_1\right|<\eta$. The theorem is proved.

3. Main Results

The main result is contained in the following theorem.

Theorem 6. *Let the conditions: H1, H2, H3, H4 and H6 hold.*
Then the solution of the problem (1), (2), (3), (4) depends continuously on the initial condition and the switching functions.

Proof. Let ε and η be the arbitrary positive constants and $T>t_0$. The following cases are posible:

Case 1. The trajectory $\gamma\left(t_0,x_0\right)$ of the problem (1), (2), (3), (4) meets no one of the switching hypersurfaces for $t_0\le t\le T$. In this case the assertation of the theorem follows from the classical theorem for continuous dependence.

Case 2. The trajectory $\gamma\left(t_0,x_0\right)$ meets only one hypersurface - Φ_1 for $t_0\le t<T$. Then it is satisfied:

2.1. According to the theorem 4 there are

$$\left(\exists\delta^i=const>0\right):\left(\forall t_0^*\in R^+,\ \left|t_0^*-t_0\right|<\delta^i\right)\left(\forall x_0^*\in D,\ \left\|x_0^*-x_0\right\|<\delta^i\right)$$

$$\left(\forall\varphi_1^* \in C[D,\square],\ \left|\varphi_1^*(x)-\varphi_1(x)\right|<\delta^i \ \ for\ x\in D\right)\ \Rightarrow\ \gamma^*\left(t_0^*,x_0^*\right)\cap\Phi_1^*\neq\varnothing,$$

i.e. the trajectory $\gamma^*\left(t_0^*,x_0^*\right)$ meets the switching hypersurface Φ_1^* at the moment t_1^*;

2.2. According to the theorem 5 it is satisfied:

$$\left(\forall\delta^{iii},\ 0<\delta^{iii}<\eta\right)\left(\exists\delta^{ii}=const>0\right):\left(\forall t_0^*\in R^+,\ \left|t_0^*-t_0\right|<\delta^{ii}\right)\left(\forall x_0^*\in D,\ \left\|x_0^*-x_0\right\|<\delta^{ii}\right)$$

$$\left(\forall\varphi_1^*\in C[D,R],\ \left|\varphi_1^*(x)-\varphi_1(x)\right|<\delta^{ii}\ \ for\ x\in D\right)\ \Rightarrow\ \left|t_1^*-t_1\right|<\delta^{iii},$$

where δ^{iii} we will specify later. From the last inequality it follows that $\left|t_1^*-t_1\right|<\eta$;

2.3. From the classical theorem for continuous dependence the following is true

$$\left(\forall\delta^{v},\ 0<\delta^{v}<\varepsilon\right)\left(\exists\delta^{iv}=const>0\right):\left(\forall t_0^*\in R^+,\ \left|t_0^*-t_0\right|<\delta^{iv}\right)\left(\forall x_0^*\in D,\ \left\|x_0^*-x_0\right\|<\delta^{iv}\right)$$

$$\Rightarrow\left\|x^*\left(t;t_0^*,x_0^*\right)-x\left(t;t_0,x_0\right)\right\|<\delta^{v}<\varepsilon,\ t_0^{\max}\le t\le t_1^{\min},$$

where $t_1^{\min}=\min\left\{t_1^*,t_1\right\}$. The constant δ^{v} we will specify later;

2.4. Let assume that $t_1^{\min}=t_1$ and $t_1^{\max}=t_1^*$. The considerations in the second case are similar. Using the condition H1 we deduce that

$$\left\|x^*\left(t_1^*;t_0^*,x_0^*\right)-x\left(t_1;t_0,x_0\right)\right\|=\left\|x^*\left(t_1;t_0^*,x_0^*\right)+\int_{t_1}^{t_1^*}f_1\left(\tau,x^*\left(\tau;t_0^*,x_0^*\right)\right)d\tau-x\left(t_1;t_0,x_0\right)\right\|$$

$$\le\delta^{v}+\int_{t_1}^{t_1^*}\left\|f_1\left(\tau,x^*\left(\tau;t_0^*,x_0^*\right)\right)\right\|d\tau\le\delta^{v}+C_f\left|t_1^*-t_1\right|=\delta^{v}+C_f\delta^{iii}.$$

2.5. From the continuity of function I_1 and taking into account the previous paragraph it follows that

$$\left(\forall\delta^{vi}=const>0\right)\left(\exists\delta^{iii}>0,\ \delta^{v}>0\right):\left\|x^*\left(t_1^*+0;t_0^*,x_0^*\right)-x\left(t_1+0;t_0,x_0\right)\right\|$$

$$\le\left\|x^*\left(t_1^*;t_0^*,x_0^*\right)-x\left(t_1;t_0,x_0\right)\right\|+\left\|I_1\left(x^*\left(t_1^*;t_0^*,x_0^*\right)\right)-I_1\left(x\left(t_1;t_0,x_0\right)\right)\right\|<\delta^{vi},$$

where the constant δ^{vi} we will determine later.

2.6. Again with the classical theorem for continuous dependence we obtain

$$\left(\exists\delta^{iii}>0\right)\left(\exists\delta^{vi}>0\right):$$

$$\left(\forall t_1^*,\ \left|t_1^*-t_1\right|<\delta^{iii}\right)\left(\forall x^*\left(t_1^*+0;t_0^*,x_0^*\right),\ \left\|x^*\left(t_1^*+0;t_0^*,x_0^*\right)-x\left(t_1+0;t_0,x_0\right)\right\|<\delta^{vi}\right)$$

$$\Rightarrow\ \left\|x^*\left(t;t_0^*,x_0^*,\varphi_1^*\right)-x\left(t;t_0,x_0,\varphi_1\right)\right\|<\varepsilon,\ t_1^{\max}\le t\le T.$$

2.7. We perform the specifying of the constants in the following sequence:

2.7.1. From 2.1 we specify δ^{i};

2.7.2. From 2.6 we specify δ^{iii} and δ^{vi};

2.7.3. From 2.5 we determine δ^{v} and futher refine δ^{iii};

2.7.4. From 2.3 we find δ^{iv};

2.7.5. From 2.2 futher refine $\delta^{iii}\left(\delta^{iii}<\eta\right)$ and determine δ^{ii}.

2.8. Let $\delta=\min\left\{\delta^{i},...,\delta^{vi}\right\}$. The obtained result can be sumarized as:

$$\left(\forall t_0^*\in R^+,\ \left|t_0^*-t_0\right|<\delta\right)\left(\forall x_0^*\in D,\ \left\|x_0^*-x_0\right\|<\delta\right)$$

$$\left(\forall\varphi_1^*\in C[D,R],\ \left|\varphi_1^*(x)-\varphi_1(x)\right|<\delta\ \ for\ x\in D\right)$$

Whence it follows:

2.8.1. $\left\| x^*\left(t;t_0^*,x_0^*\right)-x\left(t;t_0,x_0\right)\right\|<\varepsilon,\ t_0^{\max}\le t\le t_1^{\min}$ (see 2.3);

2.8.2. $\left\| x^*\left(t;t_0^*,x_0^*,\varphi_1^*\right)-x\left(t;t_0,x_0,\varphi_1\right)\right\|<\varepsilon,\ t_1^{\max}\le t\le T$ (see 2.6);

2.8.3. $\left|t_1^*-t_1\right|=\left|t_1^{\max}-t_1^{\min}\right|<\eta$ (see 2.2).

We rewrite the inequalities 2.8.1, 2.8.2 and 2.8.3 more compact in the form:

$$\left\| x^*\left(t;t_0^*,x_0^*,\varphi_1^*,\varphi_2^*,...\right)-x\left(t;t_0,x_0,\varphi_1,\varphi_2,...\right)\right\|<\varepsilon \quad for\ t_0^{\max}\le t\le T \quad and \quad \left|t-t_1\right|>\eta .$$

The theorem in this case is proved.

Case 3. The trajectory of the fundamental problem meets finite number hypersurfaces for $t_0\le t<T$. We assume that the following inequalities are fulfilled for the moments of these meetings:

$$t_0<t_1<t_2<...<t_k<T<t_{k+1}<...$$

We introduce the following notations:

$$T_0=t_0^{\max},\ T_1=\frac{1}{2}\left(t_1+t_2\right),\ T_2=\frac{1}{2}\left(t_2+t_3\right),...,T_{k-1}=\frac{1}{2}\left(t_{k-1}+t_k\right),\ T_k=T .$$

Under the previous case, we have:

3.1. $\left(\forall\delta_2,\ 0<\delta_2<\varepsilon\right)\left(\exists\delta_1,\ 0<\delta_1<\varepsilon\right)$:

$$\left(\forall t_0^*\in R^+,\ \left|t_0^*-t_0\right|<\delta_1\right)\left(\forall x_0^*\in D,\ \left\|x_0^*-x_0\right\|<\delta_1\right)$$

$$\left(\forall\varphi_1^*\in C\left[D,R\right],\ \left|\varphi_1^*\left(x\right)-\varphi_1\left(x\right)\right|<\delta_1\ \ for\ x\in D\right)$$

$$\Rightarrow\left\| x^*\left(t;t_0^*,x_0^*\right)-x\left(t;t_0,x_0\right)\right\|<\delta_2,\ T_0\le t\le t_1^{\min};$$

$$\left\| x^*\left(t;t_0^*,x_0^*,\varphi_1^*\right)-x\left(t;t_0,x_0,\varphi_1\right)\right\|<\delta_2,\ t_1^{\min}\le t\le T_1\ \ and\ \ t_1^{\max}-t_1^{\min}<\eta .$$

3.2. $\left(\forall\delta_3,\ 0<\delta_3<\varepsilon\right)\left(\exists\delta_2,\ 0<\delta_2<\varepsilon\right)$:

$$\left(\forall x^*\left(t;t_0^*,x_0^*,\varphi_1^*\right),\ \left\| x^*\left(T_1;t_0^*,x_0^*,\varphi_1^*\right)-x\left(T_1;t_0,x_0,\varphi_1\right)\right\|<\delta_2\right)$$

$$\left(\forall\varphi_2^*\in C\left[D,R\right],\ \left|\varphi_2^*\left(x\right)-\varphi_2\left(x\right)\right|<\delta_2\ \ for\ x\in D\right)$$

$$\Rightarrow\left\| x^*\left(t;t_0^*,x_0^*,\varphi_1^*\right)-x\left(t;t_0,x_0,\varphi_1\right)\right\|<\delta_3,\ T_1\le t\le t_2^{\min};$$

$$\left\| x^*\left(t;t_0^*,x_0^*,\varphi_1^*,\varphi_2^*\right)-x\left(t;t_0,x_0,\varphi_1,\varphi_2\right)\right\|<\delta_3,\ t_2^{\min}\le t\le T_2\ \ and\ \ t_2^{\max}-t_2^{\min}<\eta .$$

....

3.к-1. $\left(\forall\delta_k,\ 0<\delta_k<\varepsilon\right)\left(\exists\delta_{k-1},\ 0<\delta_{k-1}<\varepsilon\right)$:

$$\left(\forall x^*\left(t;t_0^*,x_0^*,\varphi_1^*,...,\varphi_{k-2}^*\right),\ \left\| x^*\left(T_1;t_0^*,x_0^*,\varphi_1^*,...,\varphi_{k-2}^*\right)-x\left(T_1;t_0,x_0,\varphi_1,...,\varphi_{k-2}\right)\right\|<\delta_{k-1}\right)$$

$$\left(\forall\varphi_{k-1}^*\in C\left[D,R\right],\ \left|\varphi_{k-1}^*\left(x\right)-\varphi_{k-1}\left(x\right)\right|<\delta_{k-1}\ \ for\ x\in D\right)$$

$$\Rightarrow\left\| x^*\left(t;t_0^*,x_0^*,\varphi_1^*,...,\varphi_{k-2}^*\right)-x\left(t;t_0,x_0,\varphi_1,...,\varphi_{k-2}\right)\right\|<\delta_k,\ \ T_{k-2}\le t\le t_{k-1}^{\min};$$

$$\left\| x^*\left(t;t_0^*,x_0^*,\varphi_1^*,...,\varphi_{k-1}^*\right)-x\left(t;t_0,x_0,\varphi_1,...,\varphi_{k-1}\right)\right\|<\delta_k,\ \ t_{k-1}^{\min}\le t\le T_{k-1}\ \ and\ \ t_{k-1}^{\max}-t_{k-1}^{\min}<\eta .$$

3.к. $\left(\exists\delta_k,\ 0<\delta_k<\varepsilon\right)$:

$$\left(\forall x^*\left(t;t_0^*,x_0^*,\varphi_1^*,...,\varphi_{k-1}^*\right),\ \left\|x^*\left(T_1;t_0^*,x_0^*,\varphi_1^*,...,\varphi_{k-1}^*\right)-x\left(T_1;t_0,x_0,\varphi_1,...,\varphi_{k-1}\right)\right\|<\delta_k\ \right)$$
$$\left(\forall\varphi_k^*\in C\left[D,R\right],\ \left|\varphi_k^*\left(x\right)-\varphi_k\left(x\right)\right|<\delta_k\ \ for\ x\in D\right)$$
$$\Rightarrow\left\|x^*\left(t;t_0^*,x_0^*,\varphi_1^*,...,\varphi_{k-1}^*\right)-x\left(t;t_0,x_0,\varphi_1,...,\varphi_{k-1}\right)\right\|<\varepsilon,\ \ T_{k-1}\le t\le t_k^{\min};$$
$$\left\|x^*\left(t;t_0^*,x_0^*,\varphi_1^*,...,\varphi_k^*\right)-x\left(t;t_0,x_0,\varphi_1,...,\varphi_k\right)\right\|<\varepsilon,\ \ t_k^{\min}\le t\le T_k\ \ and\ \ t_k^{\max}-t_k^{\min}<\eta\ .$$

The constants $\delta_i,\ i=1,2,...,k$, we define in the reverse order: first, we identify δ_k , after that we determine δ_{k-1} etc. Finally, we find δ_1. We substitute $\delta=\delta_1$. The results of the paragraphs 3.1 ÷ 3.к can be summarized as:

$$\left(\forall t_0^*\in R^+,\left|t_0^*-t_0\right|<\delta\right)\left(\forall x_0^*\in D,\left\|x_0^*-x_0\right\|<\delta\right)$$
$$\left(\forall\varphi_i^*\in C\left[D,R\right],\ \left\|\varphi_i^*\left(x\right)-\varphi_i\left(x\right)\right\|<\delta\ \ for\ x\in D,\ i=1,2,...\right)\ \Rightarrow$$
$$\left(\left\|x^*\left(t;t_0^*,x_0^*,\varphi_1^*,\varphi_2^*,...\right)-x\left(t;t_0,x_0,\varphi_1,\varphi_2,...\right)\right\|<\varepsilon\ for\ t_0^{\max}\le t\le T\ and\ \left|t-t_i\right|>\eta,\ i=1,2,...\right).$$

The theorem is proved in this case.

Case 4. The trajectory of the problem (1), (2), (3), (4) meets infinity many switching hypersurfaces for $t_0\le t\le T$. The following inequalities $t_0<t_1<t_2<...<T$ are valid in this case. The last inequalities contradict to the second statement of the theorem 1. Therefore, this case is impossible.
The theorem is proved.

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