IMPULSIVE DIFFERENTIAL INEQUALITIES AND COMPARISON THEOREMS FOR
POSITIVE SOLUTIONS OF NEUTRAL DELAY EQUATIONS

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Abstract

It has become imperative in recent times to determine the properties of the solutions of certain mathematical

equations or inequalities from the knowledge of associated inequalities or equations. In this paper, a number

of oscillatory properties of the solutions of impulsive differential inequalities are established using the knowledge of

those of their associated equations and what is more, some comparison theorems for the positive solutions of
certain neutral delay impulsive differential equations are also formulated.

KEY WORDS: Neutral delay impulsive differential inequalities and equations; Comparison results and theorems.

1. Introduction

Sometimes in analysis and in many other considerations, there exists the need to compare the properties of the

solutions of certain mathematical equations and inequalities. Such comparison may take the form of conclusions

about the behaviours of the solutions of the inequalities knowing the behaviours of the solutions of the associated
equations and vice versa. In this paper, we

(i) establish some comparison results for positive solutions of a neutral delay impulsive differential equation of the form

\[
\begin{align*}
  &x(t) + \sum_{j=1}^{M} p_j(t)x(t - \tau_j(t)) + \sum_{i=1}^{N} q_i(t)x(t - \sigma_i(t)) = 0, t \neq t_k \\
  &\Delta x(t_k) + \sum_{j=1}^{M} p_j(t_k)x(t_k - \tau_j(t_k)) + \sum_{i=1}^{N} q_i x(t_k - \sigma_i(t_k)) = 0
\end{align*}
\]

and the inequalities

\[
\begin{align*}
  &y(t) + \sum_{j=1}^{M} a_j(t)y(t - \tau_j(t)) + \sum_{i=1}^{N} b_i(t)y(t - \sigma_i(t)) \leq 0, t \neq t_k \\
  &\Delta y(t_k) + \sum_{j=1}^{M} a_j(t_k)y(t_k - \tau_j(t_k)) + \sum_{i=1}^{N} b_i y(t_k - \sigma_i(t_k)) \leq 0
\end{align*}
\]

and

\[
\begin{align*}
  &z(t) + \sum_{j=1}^{M} c_j(t)z(t - \tau_j(t)) + \sum_{i=1}^{N} d_i(t)z(t - \sigma_i(t_k)) \geq 0, t \neq t_k \\
  &\Delta z(t_k) + \sum_{j=1}^{M} c_j(t_k)z(t_k - \tau_j(t_k)) + \sum_{i=1}^{N} d_i z(t_k - \sigma_i(t_k)) \geq 0
\end{align*}
\]

where \( p_j, a_j, c_j, q_i, b_i \) and \( d_i \) are temporarily assumed to be piece-wise continuous functions; \( \tau_j \) and \( \sigma_i \) are

continuous and \( q_{ik}, b_{ik} \) and \( d_{ik} \) are positive constants for all \( t, t_k \in \mathbb{R}_+, 1 \leq j \leq M \) and \( 1 \leq i \leq N \); and

(ii) formulate comparison theorems for neutral delay impulsive differential equations with the aid of some

known oscillatory properties of the solutions of other given delay impulsive equations.

Unlike ordinary differential equations, the solution \( x(t) \) for \( t \in [t_0, T] \) of an impulsive differential equation or its

first derivative \( x'(t) \) is a piece-wise continuous function with points of discontinuity \( t_k \in [t_0, T], t_k \neq t \).
Therefore, in order to simplify the statements of our assertions later, we introduce the set of functions \( PC \) and \( PC' \) which are defined as follows:

Let \( r \in \mathbb{N}, \) \( D := [T, \infty) \subset \mathbb{R} \) and let \( S := \{ E \}_{k \in E} \), where \( E \) represents a subscript set which can be the set of natural numbers \( \mathbb{N} \) or the set of integers \( \mathbb{Z} \), be fixed. Throughout our discussion, we will assume that the sequences \( \{ E \} \) are moments of impulse effect and satisfy the properties:

\[ C1.1 \quad \text{If} \quad \{ E \} \quad \text{is defined with} \quad E := \mathbb{N}, \quad \text{then} \quad 0 < t_1 < t_2 < \cdots \quad \text{and} \quad \lim_{k \to \infty} t_k = +\infty; \]

\[ C1.2 \quad \text{If} \quad \{ E \} \quad \text{is defined with} \quad E := \mathbb{Z}, \quad \text{then} \quad t_0 \leq 0 < t_1, \quad t_k < t_{k+1} \quad \text{for all} \quad k \in \mathbb{Z}, \quad k \neq 0 \quad \text{and} \quad \lim_{k \to \pm \infty} t_k = \pm \infty . \]

We denote by \( PC(D, R) \) the set of all functions \( \varphi : D \to R \), which are continuous for all \( t \in D, \) \( t \not\in S \). They are continuous from the left and have discontinuity of the first kind at the points for which \( t \in S \).

By \( PC'(D, R) \), we denote the set of functions \( \varphi : D \to R \) having derivative \( \frac{d\varphi}{dt} \in PC(D, R), \) \( 0 \leq j \leq r \) ([Bainov/Simeonov: 1998]; [Lakshmikantham et al: 1989]). To specify the points of discontinuity of functions belonging to \( PC \) or \( PC' \), we shall sometimes use the symbols \( PC(D, R; S) \) and \( PC(D, R; S), \) \( r \in \mathbb{N} \). In the sequel, all functional inequalities that we write are assumed to hold finally, that is, for all sufficiently large \( t \).

Let \( \gamma = \max \{ t_j, \sigma_i \} \) and let \( t_i \geq t_0 \). By a solution of equation (1.1), we mean a function \( x(t) \in PC[[t_1 - \gamma, \infty], R] \) such that \( x(t) + \sum_{j=1}^{M} p_j(t)x(t - \tau_j(t)) \) is piece-wise continuously differentiable for \( t \geq t_1 \) and such that equation (1.1) is satisfied for all \( t \geq t_1 \).

Let \( t_i \geq t_0 \) be a given initial point and let \( \varphi \in PC[[t_1 - \gamma, t_1], R] \) be a given initial function. Then if \( x(t - \tau_j(t)) \in PC'[[t_1 - \gamma, t_1], R], \) \( x(t - \sigma_i(t)) \in PC[[t_1 - \gamma, t_1], R] \) and

\[ \lim_{t \to \infty} (t - \tau_j(t)) = \lim_{t \to \infty} (t - \sigma_i(t)) = \infty, \quad \forall t \geq t_1, \quad \tau_j, \quad \sigma_i \in \mathbb{R}_+ , \quad (1.4) \]

equation (1.1) has a unique solution on \([t_1, \infty)\) satisfying the initial condition

\[ x(t) = \varphi(t) \quad \text{for} \quad t_1 - \gamma \leq t \leq t_1 \quad (\text{Bainov/Simeonov: 1998}). \quad (1.5) \]

A solution \( x \) of the initial value problem (1.1) and (1.5) on \([t_1, \infty)\) is said to be

(i) finally positive, if there exists \( T \geq 0 \) such that \( x(t) \) is defined for \( t \geq T \) and \( x(t) > 0 \) for all \( t \geq T \);

(ii) finally negative, if there exists \( T \geq 0 \) such that \( x(t) \) is defined for \( t \geq T \) and \( x(t) < 0 \) for all \( t \geq T \);

(iii) regular, if it is defined in some half line \([T_x, \infty)\) for some \( T_x \in \mathbb{R} \) and

\[ \sup \{ x(t) : t \geq T_x \} > 0, \quad \forall T > T_x \quad (\text{Lakshmikantham et al: 1989}). \]

In what follows, we will denote by \( x(\varphi; t) \), the unique solution of the initial value problem (1.1) and (1.5). It exists throughout the interval \([t_1 - \gamma, \infty)\).

2. Statement of the Problem

The comparison results for the solutions of non-neutral delay impulsive differential equations and inequalities have been studied by many authors [{Kulenovic et al: 1990}; {Ladas et al: 1992}; {Agarwal et al: 1996}; {Belinskiy et al: 2007}; {Berezanski/Braverman: 2002}; {Domshlak et al: 2002}; {El-Morshedy/Grace: 2005}]. A comprehensive treatment of such results is given in the monograph by Bainov and Simeonov (Bainov/Simeonov: 1998). One of the most important methods of such investigations is the method of generalized characteristic equations, which is based on the idea of finding the solutions of linear impulsive systems of the form.
\[ x(t) = \varphi(t_0) \exp \left( - \int_{t_0}^{t} \alpha(s) \, ds \right) \prod_{k \, \in \, 1 \leq i \leq t} (1 - \alpha_i) , \quad t \geq t_0. \]  

(2.1)

Our main objective is to apply this method to equation (1.1) and inequalities (1.2) and (1.3) to find comparison conditions and to generalize and extend the idea for the determination of the oscillatory properties earlier referred to.

We return to equation (1.1) and based on it, introduce the conditions

\[
\begin{aligned}
\rho_i &\in \mathcal{P} \left( [t_0, \infty), R_+ \right), \quad \tau_j \in \mathcal{C} \left( [t_0, \infty), R_+ \right), \quad 1 \leq j \leq M, \\
q_i &\in \mathcal{P} \left( [t_0, \infty), R_+ \right), \quad \sigma_i \in \mathcal{C} \left( [t_0, \infty), R_+ \right), \quad 1 \leq i \leq N
\end{aligned}
\]

and

\[
\begin{aligned}
C2.1 & \quad \text{Let } t_0 \in R_+ \text{ and define } t_{-1} = \min \{ T_{-1}, \bar{T}_{-1} \}, \text{ where }
T_{-1} = \min \left\{ \inf \left\{ t - \tau_j (t) \right\} \right\} \text{ and } \bar{T}_{-1} = \min \left\{ \inf \left\{ t - \sigma_i (t) \right\} \right\}.
\end{aligned}
\]

For each \( t_0 \in R_+ \), we define the following functions:

\[
\begin{aligned}
\eta_j (t) &= \min \left\{ t_0, t - \tau_j (t) \right\}, \quad \eta_j (t) = \max \left\{ t_0, t - \tau_j (t) \right\}, \quad 1 \leq j \leq M, \\
\eta_j (t) &= \min \left\{ t_0, t - \sigma_i (t) \right\}, \quad \eta_j (t) = \max \left\{ t_0, t - \sigma_i (t) \right\}, \quad 1 \leq i \leq M.
\end{aligned}
\]

(2.2)

and write the generalized characteristic system as

\[
\begin{aligned}
\alpha(t) &= S_{pq} \left[ \alpha \right](t), \\
\alpha_k &= l_{pq} \left[ \alpha \right](k),
\end{aligned}
\]

(3.3)

where \( t \geq t_0, t_k \geq t_0 \) and \( t_k \in S \) for all \( k \in Z \); \( S_{pq}, l_{pq} \) are generalized characteristic operators acting on the neutral delay impulsive differential equations with the coefficients \( p \) and \( q \) for \( t \) and \( t_k \) respectively; and

\[
\begin{aligned}
S_{pq} \left[ \alpha \right](t) &= \Lambda_{11}(t) \prod_{k \, \in \, 1 \leq k \, \leq t} \left( 1 - \alpha_k \right)^{-1} + \Lambda_{21}(t) \prod_{k \, \in \, 1 \leq k \, \leq t} \left( 1 - \alpha_k \right)^{-1}, \\
l_{pq} \left[ \alpha \right](k) &= \Lambda_{12}(t_k) \prod_{k \, \in \, 1 \leq k \, \leq t_k} \left( 1 - \alpha_k \right)^{-1} + \Lambda_{22}(t_k) \prod_{k \, \in \, 1 \leq k \, \leq t_k} \left( 1 - \alpha_k \right)^{-1}.
\end{aligned}
\]

(2.4)

The operator elements \( \Lambda_{m'n'}(*) \), \( 1 \leq m', n' \leq 2 \), are defined by

\[
\Lambda_{11}(t) = \sum_{j=1}^{\infty} \left[ p_j (t) (\tau_j (t) - 1) \frac{\omega(h_j (t))}{\varphi'(t_0)} \right] \frac{\varphi(h_j (t))}{\varphi(t_0)} \exp \left( \int_{h_j (t)}^{t} \alpha(s) \, ds \right),
\]

\[
\Lambda_{12}(t_k) = \sum_{j=1}^{\infty} \left[ p_j (t_k) (\Delta \tau_j (t_k) - 1) \frac{\Delta \varphi(h_j (t_k))}{\varphi'(t_0)} \right] \frac{\varphi(h_j (t_k))}{\varphi(t_0)} \exp \left( \int_{h_j (t_k)}^{t} \alpha(s) \, ds \right),
\]

\[
\Lambda_{21}(t) = \sum_{k=1}^{\infty} \left[ q_k (t) \frac{\varphi(g_k (t))}{\varphi(t_0)} \right] \exp \left( \int_{g_k (t)}^{t} \alpha(s) \, ds \right)
\]

and

\[
\Lambda_{22}(t_k) = \sum_{k=1}^{\infty} \left[ q_k (t_k) \frac{\varphi(g_k (t_k))}{\varphi(t_0)} \right] \exp \left( \int_{g_k (t_k)}^{t} \alpha(s) \, ds \right), \quad \forall t > t_0.
\]

Let \( J \subset R \) be a given interval and \( \Omega(J) \) a set of piece-wise continuous functions defined in \( J \). In future, the notation \( (\alpha, \{\alpha_k\}_{k=1}^{\infty}) \in \Omega(J) \), where \( \alpha(t) \) is the value of \( \alpha \) at \( t \), will mean that the function \( \alpha \) and the sequence \( \{\alpha_k\}_{k=1}^{\infty} \) satisfy the condition

\[
\alpha \in \mathcal{P} \left( J, R_+ \right), \quad \alpha_k < 1 \quad \forall k \in Z \quad \text{such that} \quad t_k \in [t_0, \infty) \cap S.
\]

Now let equation (1.4) and condition

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\[ \tau_j \in C^1(\mathbb{R}_+, \mathbb{R}_+), \quad \sigma_i \in C(\mathbb{R}_+, \mathbb{R}_+), \]

be fulfilled. Further, let the coefficients of equation (1.1) be non-negative, that is,

\[ p_j \in PC^1([t_0, \infty), \mathbb{R}_+), \quad q_i \in PC([t_0, \infty), \mathbb{R}_+), \]

\[ q_{ik} \geq 0 \text{ for } k \in \mathbb{N}, 1 \leq j \leq M \text{ and } 1 \leq i \leq N. \]

Define

\[ \overline{M} = \max \{M, N\}, \]

and let \( x(t) \) be a finally positive solution of equation (1.1).

**Remark 2.1 (Equivalent of Remark 16.2 by Bainov and Simeonov ([Bainov/Simeonov: 1998]))**

Assume that condition C2.3 holds. Then for sufficiently large \( t, h_j(t) = t_0, H_j(t) = t - \tau_j(t), q_j(t) = t_0 \) and \( G_j(t) = t - \sigma_i(t) \) for \( 1 \leq j \leq M, 1 \leq i \leq N \) and \( \forall t \in [t_0, T) \) the system (2.3) transforms into the equation

\[
\begin{align*}
\alpha(t) & = \sum_{j=1}^{M} p_j(t)(\tau_j(t) - 1) \alpha(t - \tau_j(t)) + p_j(t) \exp \left( \int_{t - \tau_j(t)}^{1} \alpha(s) \, ds \right) \prod_{1 \leq \tau_j(t) < t} (1 - \alpha) + \\
& \quad + \sum_{i=1}^{N} q_i(t) \exp \left( \int_{t - \sigma_i(t)}^{1} \alpha(s) \, ds \right) \prod_{t - \sigma_i(t) < t} (1 - \alpha),
\end{align*}
\]

\[ (2.5) \]

\[
\alpha_k = \sum_{j=1}^{M} \left[ p_j(t_k) (\Delta \tau_j(t_k) - 1) \alpha(t_k - \tau_j(t_k)) + \Delta p_j(t_k) \right] \exp \left( \int_{t_k - \tau_j(t_k)}^{b_k} \alpha(s) \, ds \right) \prod_{t_k - \tau_j(t_k) < t_k} (1 - \alpha),
\]

\[ + \sum_{i=1}^{N} q_k(t) \exp \left( \int_{t_k - \sigma_i(t_k)}^{b_k} \alpha(s) \, ds \right) \prod_{t_k - \sigma_i(t_k) < t_k} (1 - \alpha). \]

**Corollary 2.1 (Equivalent of Corollary 16.2 by Bainov and Simeonov ([Bainov/Simeonov: 1998]))**

Let conditions C2.1 and C2.3 be fulfilled. Suppose equation (1.1) has a finally positive solution \( x(t) \) for \( t \in [t_0, T) \) and \( x(t^+) > 0 \) \( \forall t \in \mathbb{R}_+ \) such that \( t_k \in \mathbb{S} \). Then there exist \( t_1 > 0 \) and \( (\alpha, \{\alpha_k\}_{k=1}^{\infty}) \in \Omega([t_1, + \infty)) \) such that \( (\alpha, \{\alpha_k\}_{k=1}^{\infty}) \) is a solution of equation (2.4) for \( t > t_1 \geq 0 \).

Now, let condition C2.3 be fulfilled and suppose that the coefficients of equation (1.1) are nonnegative, that is, condition C2.4 is satisfied. Let \( x \) be a finally positive solution of equation (1.1), and let the function \( \alpha \) and the sequence \( \{\alpha_k\}_{k=1}^{\infty} \) be defined as

\[
\alpha(t) = -\frac{x'(t)}{x(t)}, \quad \alpha_k = -\frac{\Delta x(t_k)}{x(t_k)},
\]

\[ (2.6) \]

for \( t \in [t_0, T) \) and \( \forall k \in \mathbb{N}, t_k \in \mathbb{S} \). Then from Corollary 2.1, there exists \( t_1 > 0 \) such that \( (\alpha, \{\alpha_k\}_{k=1}^{\infty}) \) is a solution of equation (2.5) for \( t \geq t_0 \) and what is more,

\[
\sum_{\xi=1}^{M} p_{\xi}(t) + q_{\xi}(t) \leq \alpha(t), \sum_{\xi=1}^{M} \left[ \Delta p_{\xi}(t_k) + q_{\xi}(t) \right] \leq \alpha_k < 1, \quad x(t^+) > 0
\]

\[ (2.7) \]

for \( t \geq t_1, t_k \geq t_1, 1 \leq k < \infty, 1 \leq \xi \leq \overline{M}. \)

Let \( t_{-1} \) be as defined in condition C2.2, \( J := [t_{-1}, + \infty) \) and set

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\[
\left\{ \begin{array}{l}
\mathcal{S}_{pq}[\lambda]_{t} = \sum_{j=1}^{M} \left[ p_{j}(t)\left(\tau_{j}(t) - t\right) + \frac{1}{\gamma_{j}(t)} \right] \exp\left( \int_{t-\tau_{j}(t)}^{t} \lambda(s)ds \right) \prod_{j=1}^{M} (1-\lambda_{j})^{-1} + \\
+ \sum_{i=1}^{N} q_{i}(t) \exp\left( \int_{t}^{\tau_{i}(t)} \lambda(s)ds \right) \prod_{i=1}^{N} (1-\lambda_{i})^{-1}, \\
\end{array} \right.
\]

\[\tilde{\mathcal{T}}_{pq}[\lambda](k) = \sum_{j=1}^{M} \left[ p_{j}(t_{k})\left(\Delta\tau_{j}(t_{k}) - 1\right) + \Delta p_{j}(t_{k}) \right] \exp\left( \int_{t_{k}-\tau_{j}(t_{k})}^{t_{k}} \alpha(s)ds \right) \prod_{j=1}^{M} (1-\alpha_{j})^{-1} + \\
+ \sum_{i=1}^{N} q_{i} \exp\left( \int_{t_{k}-\tau_{i}(t_{k})}^{t_{k}} \alpha(s)ds \right) \prod_{i=1}^{N} (1-\alpha_{i})^{-1}
\] (2.8)

for \( (\lambda, \{\lambda_{k}\}_{k=1}^{\infty}) \in \Omega(J) \) and \( t \geq t_{1}, t_{k} \geq t_{1}, 1 \leq k < \infty \). Then:

(i) \( \tilde{\mathcal{S}}_{pq}[\lambda](t) = \sum_{j=1}^{M} \left[ p_{j}(t) + \frac{1}{\gamma_{j}(t)} \right] \), \( \tilde{\mathcal{T}}_{pq}[\lambda](k) = \sum_{j=1}^{M} \left[ \Delta p_{j}(t_{k}) + q_{j}(t) \right] \), \( t \geq t_{1}, t_{k} \geq t_{1} \) and \( t_{k} \in S \);

(ii) if \( \left( \beta, \{\beta_{k}\}_{k=1}^{\infty} \right) \in \Omega(J) \), \( \left( \gamma, \{\gamma_{k}\}_{k=1}^{\infty} \right) \in \Omega(J) \) and \( \beta(t) \leq \gamma(t) \), \( \beta_{k} \leq \gamma_{k} \), \( t \geq t_{-1} \), \( k \in N \)

then \( \tilde{\mathcal{S}}_{pq}[\beta](t) \leq \tilde{\mathcal{S}}_{pq}[\gamma](t), \ t \in S \),

\[ \tilde{\mathcal{T}}_{pq}[\beta](k) \leq \tilde{\mathcal{T}}_{pq}[\gamma](k), \forall t_{k} \in S. \]

To each \( t_{1} > 0 \) and \( t_{-1} \) as defined in condition C2.2, we relay the following inductively defined functional and numerical sequences:

\[
\left\{ \begin{array}{l}
u_{0}(t) = 0, \text{ for } t \geq t_{-1} \\
u_{0k} = 0, \text{ for } t_{k} \geq t_{-1} \\
u_{m+1}(t) = \tilde{\mathcal{S}}_{pq}[\nu_{m}](t), t \geq t_{1}, \\
u_{m+1k} = \tilde{\mathcal{T}}_{pq}[\nu_{m}](k), t_{k} \geq t_{1}, t_{k} \in S.
\end{array} \right.
\] (2.9)

**Theorem 2.1 (Equivalent of Theorem 16.2 by Bainov and Simeonov ([Bainov/Simeonov: 1998]))**

Let conditions C2.3 and C2.4 hold. Then the following statements are equivalent:

(a) equation (1.1) has a finally positive solution;

(b) there exists \( t_{1} \geq 0 \) such that the sequences \( \{u_{m}\} \) and \( \{u_{mk}\} \), defined by equation (2.9) converge pointwise and monotonically, that is,

\[
\lim_{m \to +\infty} u_{m}(t) = u(t), \ t \geq t_{1} \\
\lim_{m \to +\infty} u_{mk} = u_{k}, \ t_{k} \geq t_{1}, t_{k} \in S
\] (2.10)

and

\[ u_{k} < 1 \text{ for all } t_{k} \geq t_{1} \text{ and } t_{k} \in S. \] (2.11)

**3. Main Results**

Let \( t_{0} \geq 0 \) and \( t_{-1} \) as defined in condition C2.2. We recall the neutral delay impulsive differential equation (1.1) and the inequalities (1.2) and (1.3) and introduce the conditions

C3.1 \[ \left\{ \begin{array}{l}
p_{j}, a_{j}, c_{j} \in \mathcal{PC}(\mathcal{R}_{+}, \mathcal{R}_{+}), \ q_{j}, b_{j}, d_{j} \in \mathcal{PC}(\mathcal{R}_{+}, \mathcal{R}_{+}), \ \tau_{j} \in \mathcal{C}(\mathcal{R}_{+}, \mathcal{R}_{+}) \\
\sigma_{j} \in \mathcal{C}(\mathcal{R}_{+}, \mathcal{R}_{+}), \ q_{k}, b_{k}, d_{k} \geq 0, \ k \in N, \ 1 \leq j \leq M \text{ and } 1 \leq i \leq N,
\end{array} \right. \]
and

$$C3.2\quad \begin{cases} a_i(t) \geq p_i(t) \geq c_i(t), & b_i(t) \geq q_i(t) \geq d_i(t), & \forall t \in \mathbb{R}_+, \\
 b_k \geq q_k \geq d_k, & k \in \mathbb{N}, \quad 1 \leq j \leq M \text{ and } 1 \leq i \leq N.\end{cases}$$

Also, let us now suppose that the initial interval associated with the said equation and inequalities under the above conditions is $[t_{-1}, t_0]$.

**Theorem 3.1**

Let conditions C3.1 and C3.2 be fulfilled. Assume that $x$, $y$ and $z$ belong to the space $PC([t_{-1}, +\infty), \mathbb{R})$ and are solutions of equation (1.1) and inequalities (1.2) and (1.3) respectively. Suppose further, that

$$y(t) > 0, \quad t \geq t_0, \quad (3.1)$$

$$z(t^+) \geq x(t^+) \geq y(t^+), \quad (3.2)$$

$$y(t) \geq \frac{x(t)}{y(t)} \geq \frac{z(t)}{y(t)} \geq 0, \quad t_{-1} \leq t \leq t_0. \quad (3.3)$$

Then

$$z(t) \geq x(t) \geq y(t), \quad \forall t \geq t_0. \quad (3.4)$$

**Proof**

It follows from inequalities (3.1) and (3.2) that the functions $x$ and $z$ are positively defined in some right neighbourhood $[t_0, T_1)$ of $t_0$. We shall show that $T_1 = +\infty$. If we assume on the contrary, then there exists $T_1 > t_0$ such that

$$x(t) > 0, \quad z(t) > 0, \quad t_0 \leq t < T_1 \quad (3.5)$$

and

$$x(T_1^+) \leq 0 \quad \text{or} \quad z(T_1^+) \leq 0. \quad (3.6)$$

For $t \in [t_0, T_1)$ and $t_k \in [t_0, T_1) \cap S$, we set

$$\begin{cases} \alpha_0(t) = \frac{y(t)}{y(t)}, & \beta_0(t) = -\frac{x(t)}{x(t)}, & \gamma_0(t) = -\frac{z(t)}{z(t)}, \\
 \alpha_{0k} = -\frac{\Delta y(t_k)}{y(t_k)}, & \beta_{0k} = -\frac{\Delta x(t_k)}{x(t_k)}, & \gamma_{0k} = -\frac{\Delta z(t_k)}{z(t_k)}, \end{cases}$$

then

$$\alpha_0(t) \geq S_{ab}[\alpha_0](t), \quad \alpha_{0k} \geq l_{ab}[\alpha_0](k), \quad (3.7)$$

$$\beta_0(t) = S_{ab}[\beta_0](t), \quad \beta_{0k} = l_{ab}[\beta_0](k), \quad (3.8)$$

$$\gamma_0(t) \leq S_{cd}[\gamma_0](t), \quad \gamma_{0k} \leq l_{cd}[\gamma_0](k) \quad (3.9)$$

for $t \in [t_0, T_1) \cap S$, $t_k \in S$, where

$$S_{ab}[\alpha](t) = \sum_{j=1}^{N} \left[ a_j(t)(\alpha_j(t) - 1) \frac{\varphi(h_j(t))}{\varphi((t_0))} \alpha_j(h_j(t)) + a'_j(t) \frac{\varphi(h_j(t))}{\varphi((t_0))} \right] x$$

$$l_{ab}[\alpha](k) = \sum_{j=1}^{M} \left[ a_j(t_k)(\Delta \alpha_j(t_k) - 1) \frac{\Delta \varphi(h_j(t_k))}{\Delta \varphi((t_0))} \alpha_j(h_j(t_k)) + \Delta a_j(t_k) \frac{\varphi(h_j(t_k))}{\varphi((t_0))} \right] x$$
\[ S_{pq}[\beta](t) = \sum_{j=1}^{M} \left[ p_j(t)(\tau_j(t) - t) \frac{\phi(h_j(t))}{\phi((t_0))} \beta(H_j(t)) + p'_j(t) \frac{\phi(h_j(t))}{\phi((t_0))} \right] x \times \left[ \int_{H_j(t)}^{\beta(s)ds} \prod_{j \neq i, < t}^{N} (1 - \beta_{j,i})^{-1} + \sum_{i=1}^{N} q_i(t) \frac{\phi(g_i(t))}{\phi((t_0))} \right] \prod_{j \neq i, < t}^{N} (1 - \beta_{j,i})^{-1} \]

\[ l_{pq}[\beta](k) = \sum_{j=1}^{M} \left[ p_j(t) \left( \Delta \tau_j(t) - t \right) \frac{\Delta \phi(h_j(t))}{\Delta \phi((t_0))} \beta(H_j(t)) + \Delta p_j(t) \frac{\phi(h_j(t))}{\phi((t_0))} \right] x \times \left[ \int_{H_j(t)}^{\beta(s)ds} \prod_{j \neq i, < t}^{N} (1 - \beta_{j,i})^{-1} + \sum_{i=1}^{N} q_i(t) \frac{\phi(g_i(k))}{\phi((t_0))} \right] \prod_{j \neq i, < t}^{N} (1 - \beta_{j,i})^{-1} \]

\[ S_{cd}[\gamma](t) = \sum_{j=1}^{M} \left[ c_j(t)(\tau_j(t) - t) \frac{\phi(h_j(t))}{\phi'(h_j(t))} \gamma(H_j(t)) + c'_j(t) \frac{\phi(h_j(t))}{\phi'(h_j(t))} \right] x \times \left[ \int_{H_j(t)}^{\gamma(s)ds} \prod_{j \neq i, < t}^{N} (1 - \gamma_{j,i})^{-1} + \sum_{i=1}^{N} d_i(t) \frac{\phi(g_i(t))}{\phi((t_0))} \right] \prod_{j \neq i, < t}^{N} (1 - \gamma_{j,i})^{-1} \]

\[ l_{cd}[\gamma](k) = \sum_{j=1}^{M} \left[ c_j(t) \left( \Delta \tau_j(t) - t \right) \frac{\Delta \phi(h_j(t))}{\Delta \phi((t_0))} \gamma(H_j(t)) + \Delta c_j(t) \frac{\phi(h_j(t))}{\phi((t_0))} \right] x \times \left[ \int_{H_j(t)}^{\gamma(s)ds} \prod_{j \neq i, < t}^{N} (1 - \gamma_{j,i})^{-1} + \sum_{i=1}^{N} d_i(t) \frac{\phi(g_i(k))}{\phi((t_0))} \right] \prod_{j \neq i, < t}^{N} (1 - \gamma_{j,i})^{-1} \]

and the functions \( h_j, H_j, g_i \) and \( G_i \) are as defined in the functions (2.2).

We shall prove that
\[
\begin{align*}
\alpha_0(t) &\geq \beta_0(t), t_0 \leq t < T_1 \\
\alpha_{0k} &\geq \beta_{0k}, t_0 \leq t \leq t_k < T_1, t_k \in S.
\end{align*}
\]

However, in doing this, we shall only restrict ourselves to the proof of
\[
\alpha_0(t) \geq \beta_0(t), \alpha_{0k} \geq \beta_{0k}, t_0 \leq t < T_1 \text{ and } t_k \in S,
\]

since the proof of
\[
\beta_0(t) \geq \gamma_0(t), \beta_{0k} \geq \gamma_{0k}, t_0 \leq t < T_1 \text{ and } t_k \in S
\]
is analogous and is omitted.

It follows from conditions (3.1) and (3.5) that \( (\alpha_0, (\alpha_{0k})_{k=1}^{\infty}), (\beta_0, (\beta_{0k})_{k=1}^{\infty}) \) and \( (\gamma_0, (\gamma_{0k})_{k=1}^{\infty}) \) belong to \( \Omega([t_0, T_1]) \), that is, \( \alpha_0, \beta_0, \gamma_0 \in PC([t_0, T_1], R) \) and \( \alpha_{0k}, \beta_{0k}, \gamma_{0k} < 1 \) for \( \forall k : t_k \in [t_0, T_1) \cap S \).

On the other hand, equation (3.8) suggests that \( (\beta_0, (\beta_{0k})_{k=1}^{\infty}) \) is a solution of the system
\[
\begin{align*}
\beta(t) &= S_{pq}[\beta](t) \\
\beta_k &= l_{pq}[\beta](k)
\end{align*}
\]

\( \forall t \in [t_0, T_1) \). Therefore,
\[
\begin{align*}
\beta_0(t) &\equiv \beta(t) \leq \alpha_0(t), t \in [t_0, T_1), \\
\beta_{0k} &\equiv \beta_k \leq \alpha_{0k}, t_k \in [t_0, T_1) \cap S,
\end{align*}
\]

and this is a proof of inequalities (3.11).

Obviously,
\[
y(t) = y(T_0^+) \exp \left( -\int_{t_0}^{T_0^+} \alpha_0(s)ds \right) \prod_{t_0 < t < t}^{1 - \alpha_0(t)},
\]
\[
x(t) = x(T_0^+) \exp\left(-\int_{t_0}^t \beta_0(s) ds\right) \prod_{t_0 < t_i < t} (1 - \beta_{0i})
\]
\[
z(t) = z(T_0^+) \exp\left(-\int_{t_0}^t \gamma_0(s) ds\right) \prod_{t_0 < t_i < t} (1 - \gamma_{0i})
\]

Therefore, from inequalities (3.2) and (3.10) it follows that
\[
z(t) \geq x(t) \geq y(t), \quad t_0 \leq t < T_1.
\]
Hence, from inequality (3.14) and the left-continuity of the functions \( y, x \) and \( z \),
\[
z(T_1) \geq x(T_1) \geq y(T_1) > 0
\]
which is contradictory to inequalities (3.6) for \( T_1 \not\in S \).

If \( T_1 = t_m \) for some \( m \in \mathbb{N} \), then equation (3.8) and inequalities (3.7), (3.9) and (3.10) suggest that
\[
1 > \alpha_{0m} \geq \beta_{0m} \geq \gamma_{0m}
\]
This implies that \( x(T_1) > 0 \) and \( z(T_1) > 0 \), which further confirms the contradiction of inequalities (3.6). Consequently, relation (3.4) holds for each \( t \geq t_0 \). This completes the proof of Theorem 3.1.

The following are some corollaries arising from the proved theorem and are generally important, especially in establishing linearized oscillation theorems for neutral impulsive equations.

**Corollary 3.1**

Let the following conditions hold:

\[
\begin{align*}
\mathbf{C3.3} & \quad \left\{ a_i \in \text{PC}^i(R_+, R_+), \quad b_i \in \text{PC}(R_+, R_+), \quad \tau_i \in \text{C}^i(R_+, R_+), \quad \\
& \quad \sigma_i \in \text{C}(R_+, R_+), \quad b_k \geq 0, \quad k \in \mathbb{N}, \quad 1 \leq j \leq M \quad \text{and} \quad 1 \leq i \leq N,
\end{align*}
\]

Let \( t_0 \geq 0 \) and \( \varphi, \psi \in \text{PC}([t_0-t_0], R) \) be such that
\[
\psi(t_0) \geq \varphi(t_0) > 0, \quad \frac{\varphi(t)}{\varphi(t_0)} \geq \frac{\psi(t)}{\psi(t_0)} \geq 0, \quad t_0 \leq t \leq t_0
\]
and
\[
\begin{align*}
x(\varphi; t) > 0, \quad t \geq t_0.
\end{align*}
\]

Then
\[
x(\psi; t) \geq x(\varphi; t), \quad t \geq t_0.
\]

Now consider the neutral impulsive differential inequality (1.2) together with the equation
\[
\begin{align*}
& \left[ y(t) + \sum_{i=1}^{M} a_i(t)y(t-\tau_i(t)) \right]' + \sum_{i=1}^{N} b_i(t)y(t-\sigma_i(t)) = 0, \quad t \not\in S \\
& \Delta \left[ y(t_k) + \sum_{i=1}^{M} a_i(t_k)y(t_k-\tau_i(t_k)) \right] + \sum_{i=1}^{N} b_k y(t_k-\sigma_i(t_k)) = 0, \quad \forall t_k \in S.
\end{align*}
\]

From Theorem 3.1, we immediately obtain the following result.

**Corollary 3.2**

Let conditions C3.3 be fulfilled. Then the following statements are equivalent:
(a) The neutral impulsive inequality (1.2) has a finally positive solution;
(b) Equation (3.16) has a finally positive solution.

Comparison results can also be extended to impulsive differential equations and inequalities with advanced arguments by way of adapting the proofs from the present section. For example, the following results are analogous to Corollary 3.2.

**Corollary 3.3**

Let conditions C3.3 be fulfilled. Then the following statements are equivalent:
(a) The inequality
$$\left\{ \begin{array}{l}
(x(t) + \sum_{i=1}^{M} p_{i}(t)x(t - \tau_{i}(t)))' - \sum_{i=1}^{N} q_{i}(t)x(t - \sigma_{i}(t)) \geq 0, t \notin S \\
\Delta x(t_{k}) + \sum_{i=1}^{M} p_{i}(t_{k})x(t_{k} - \tau_{i}(t_{k})) - \sum_{i=1}^{N} q_{i}(t_{k})x(t_{k} - \sigma_{i}(t_{k})) \geq 0, \forall t_{k} \in S
\end{array} \right.$$ 

has a finally positive solution

(b) The equation

$$\left\{ \begin{array}{l}
(x(t) + \sum_{i=1}^{M} p_{i}(t)x(t - \tau_{i}(t)))' - \sum_{i=1}^{N} q_{i}(t)x(t - \sigma_{i}(t)) = 0, t \notin S \\
\Delta x(t_{k}) + \sum_{i=1}^{M} p_{i}(t_{k})x(t_{k} - \tau_{i}(t_{k})) - \sum_{i=1}^{N} q_{i}(t_{k})x(t_{k} - \sigma_{i}(t_{k})) = 0, \forall t_{k} \in S
\end{array} \right.$$ 

has a finally positive solution.

Let us now return to the neutral delay impulsive differential equation (3.16) and together with it, consider the equation

$$\left\{ \begin{array}{l}
y(t) + \sum_{i=1}^{M} p_{i}(t)y(t - \mu_{i}(t)) = 0, t \notin S \\
\Delta y(t_{k}) + \sum_{i=1}^{M} p_{i}(t_{k})y(t_{k} - \mu_{i}(t_{k})) = 0, \forall t_{k} \in S.
\end{array} \right.$$ (3.17)

We introduce the following conditions:

**C3.4**

$$a_{i} \in \text{PC}^{1}(R_{+}, R_{+}), b_{i} \in \text{PC}(R_{+}, R_{+}), \tau_{i} \in C^{1}(R_{+}, R_{+}), \alpha_{i} \in C(R_{+}, R_{+}), b_{k} \geq 0, k \in N,$$

$$\lim_{t \to +\infty} [t - \tau_{i}(t)] = \lim_{t \to +\infty} [t - \sigma_{i}(t)] = +\infty, 1 \leq j \leq M, 1 \leq i \leq N$$ (3.18)

and

**C3.5**

$$p_{i} \in \text{PC}^{1}(R_{+}, R_{+}), q_{i} \in \text{PC}(R_{+}, R_{+}), \mu_{i} \in C^{1}(R_{+}, R_{+}), \lambda_{j} \in C(R_{+}, R_{+}), q_{k} \geq 0, k \in N,$$

$$\lim_{t \to +\infty} [t - \mu_{i}(t)] = \lim_{t \to +\infty} [t - \lambda_{j}(t)] = +\infty, 1 \leq j \leq M, 1 \leq i \leq N.$$ (3.19)

**Theorem 3.2**

Suppose conditions C3.4 and C3.5 are fulfilled. Let, for each $1 \leq j \leq M, 1 \leq i \leq N$,

$$a_{i}(t) \geq p_{i}(t), b_{i}(t) \geq q_{i}(t), b_{k} \geq q_{k}, t_{i}(t) \geq \mu_{i}(t),$$

$$\sigma_{i}(t) \geq \lambda_{i}(t), t \in R_{+} \forall k \in N.$$ (3.20)

Suppose further, that each regular solution of equation (3.17) is oscillatory. Then each regular solution of equation (3.16) is oscillatory.

**Proof**

For each $t_{0} \geq 0$, we set $t_{-1} = \min\{T_{-1}, \tilde{T}_{-1}\}$ and $t_{-1}' = \min\{\tilde{T}_{-1}', \tilde{T}_{-1}'\}$, where

$$T_{-1} = \min_{t \in [t_{0}, M_{1}]} \inf_{k \in I_{0}} [t - \tau_{i}(t)] \quad \tilde{T}_{-1} = \min_{t \in [t_{0}, M_{1}]} \inf_{k \in I_{0}} [t - \sigma_{i}(t)]$$

and

$$T_{-1}' = \min_{t \in [t_{0}, M_{1}]} \inf_{k \in I_{0}} [t - \mu_{i}(t)] \quad \tilde{T}_{-1}' = \min_{t \in [t_{0}, M_{1}]} \inf_{k \in I_{0}} [t - \lambda_{i}(t)],$$

and define the sequences $\{v_{m}(t)\}$ and $\{v_{mk}\}$ as follows:

$v_{0}(t) \equiv 0, v_{0k} \equiv 0$, for $t \geq t_{-1}, t_{k} \geq t_{-1}$ and $t_{k} \in S$.
\[ v_{m+1}(t) = \begin{cases} 0, & t_{-1} \leq t < t_0, \\ \sum_{j=1}^M \left[ a_j(t)(t_j(t) - 1)v_m(t - t_j(t)) + a_j'(t) \right] \exp \left( \int_{t_j(t)}^t v_m(s) \, ds \right) \prod_{t_j(t) \leq s < t} (1 - v_{m_r})^{-1} + \\ + \sum_{i=1}^N b_i(t) \exp \left( \int_{t_{-\alpha}(t)}^t v_m(s) \, ds \right) \prod_{t_{-\alpha}(t) \leq s < t} (1 - v_{m_r})^{-1}, & t \geq t_0 \end{cases} \]

\[ v_{m+1_k} = \begin{cases} 0, & t_{-1} \leq t < t_0, \\ \sum_{j=1}^M \left[ a_j(t_k)(\Delta t_j(t) - 1)v_m(t_k - t_j(t_k)) + \Delta a_j(t_k) \right] \exp \left( \int_{t_j(t_k)}^{t_k} v_m(s) \, ds \right) \prod_{t_j(t_k) \leq s < t_k} (1 - v_{m_r})^{-1} + \\ + \sum_{i=1}^N b_i(t_k) \exp \left( \int_{t_{-\alpha}(t_k)}^{t_k} v_m(s) \, ds \right) \prod_{t_{-\alpha}(t_k) \leq s < t_k} (1 - v_{m_r})^{-1}, & t_k \geq t_0, \forall t_k \in S. \end{cases} \]

Furthermore, we define the sequences \( \{u_m(t)\} \) and \( \{u_{mk}\} \) as follows:

\[ u_0(t) = 0, \ u_{bk} = 0, \text{ for } t \geq t_{-1}, \ k \geq t_{-1} \text{ and } t_k \in S \]

\[ u_{m+1}(t) = \begin{cases} 0, & t_{-1} \leq t < t_0, \\ \sum_{j=1}^M \left[ p_j(t)(t_j(t) - 1)u_m(t - t_j(t)) + p_j'(t) \right] \exp \left( \int_{t_j(t)}^t u_m(s) \, ds \right) \prod_{t_j(t) \leq s < t} (1 - u_{m_r})^{-1} + \\ + \sum_{i=1}^N q_i(t) \exp \left( \int_{t_{-\lambda_i}(t)}^t u_m(s) \, ds \right) \prod_{t_{-\lambda_i}(t) \leq s < t} (1 - u_{m_r})^{-1}, & t \geq t_0 \end{cases} \]

\[ u_{m+1_k} = \begin{cases} 0, & t_{-1} \leq t < t_0, \forall t_k \in S. \\ \sum_{j=1}^M \left[ p_j(t_k)(\Delta t_j(t) - 1)u_m(t_k - t_j(t_k)) + \Delta p_j(t_k) \right] \exp \left( \int_{t_j(t_k)}^{t_k} u_m(s) \, ds \right) \prod_{t_j(t_k) \leq s < t_k} (1 - u_{m_r})^{-1} + \\ + \sum_{i=1}^N q_i(t_k) \exp \left( \int_{t_{-\lambda_i}(t_k)}^{t_k} u_m(s) \, ds \right) \prod_{t_{-\lambda_i}(t_k) \leq s < t_k} (1 - u_{m_r})^{-1}, & t_k \geq t_0, \forall t_k \in S \end{cases} \]

For purposes of contradiction, let us assume that equation (3.16) has a non-oscillatory solution, which, in view of its linearity, may be finally positive. Consequently, from Theorem 2.1, there exists \( t_0 \geq 0 \) such that the sequences \( \{v_m(t)\} \) and \( \{v_{mk}\} \) defined by the formulas above, converge point-wise and monotonically as follows:

\[ \lim_{m \to +\infty} v_m(t) = v(t), \ t \geq t_0, \ t \not\in S. \]

\[ \lim_{m \to +\infty} v_{mk} = v_k, \ t_k \geq t_0, \forall t_k \in S. \] (3.21)

and

\[ v_k < 1 \text{ for } t_k \geq t_0 \text{ and } t_k \in S. \]

It follows from the conditions in inequalities (3.20) that

\[ u_m(t) \leq v_m(t), \ u_{mk} \leq v_{mk}. \] (3.22)

for each \( t \geq t_0, \ t_k \geq t_0 \) and \( k \in \mathbb{N} \).
On the other hand, since the sequences \( \{u_m(t)\} \) and \( \{u_{mk}\} \) are non-decreasing in respect of \( m \in \mathbb{N} \), then inequalities (3.22) imply that the limits

\[
\begin{align*}
\lim_{m \to +\infty} u_m(t) &= u(t), \quad t \geq t_0, \quad t \notin S, \\
\lim_{m \to +\infty} u_{mk} &= u_k, \quad t_k \geq t_0, \quad \forall t_k \in S.
\end{align*}
\] (3.23)

exist and

\[ u_k < 1 \quad \text{for} \quad t_k \geq t_0 \quad \text{and} \quad t_k \in S. \]

Thus, from Theorem 2.1, equation (3.17) has a finally positive solution which is a contradiction. This completes the proof of Theorem 3.2.

REFERENCES


