Variable Structure Controller for Stabilizing Delayed Systems: A Lyapunov-Krasovskii Design Approach

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Abstract

A Variable Structure Controller (VSC) is designed for the purpose of stabilizing a Time Delay System (TDS). The considered system is delay-dependent stable. The system remains stable as delay is increased from zero to a maximum value at which stability is then lost. Using a memoryless stabilization feedback, a sliding surface is constructed through a Lyapunov-Krasovskii scheme so that stability is guaranteed during sliding motion. Once the VSC is applied, simulated results are given and it is shown that sliding mode is achieved for any feasible value of the delay and initial conditions.

Key Words : Delay dependent Stability, Delay independent Stability, Lyapunov-Krasovskii Functional, Sliding Surface, Riccati Equation, Time Delay Systems, Variable Structure Controller.

1. Introduction

The issue of Variable Structure Control (VSC) of a Linear Time Invariant System (LTI) with a single constant delay in the states, is the focus of this work.The existence of delay in most systems is a source of instability and makes system analysis and control design more complicated. Often, delay is a source of poor performances.
Such a controller has the intrinsic feature of insensitivity to external disturbances (disturbance rejection property) and to plant parameter uncertainties (robustness property). In this approach, a sliding surface is constructed through a Lyapunov-Krasovskii approach. This is a natural extension of the classical Lyapunov theory in (Wu *et al.*, 1996), to delayed systems.

This paper is organized as follows. Introduction is Section 1. Section 2 reviews some milestones in the literature, as far as the application of the VSC to TDS is concerned. In Section 3,we recall some mathematical tools and bring about the Lyapunov-Krasovskii approach for delay-independent stability criterion. Section 4 is the description of the system, meanwhile Section 5 tells about the VSC design methodology and the statement of the existence conditions. The stability of the sliding mode motion is also addressed in section 5. Section 6 is for results: plots of the states and the control law are provided in the simulation. Finally, section 7 is conclusion.

2. Literature Review and Background

Disturbance rejection property and robustness to plant parameter uncertainties, are the two main advantages of VSC over classical adaptive controllers. These two features are widely reported in the literature by Drazenovic (1969), Zinober (1981), Young *et al*. (1996), Young *et al*. (1999) and Utkin (1983) and many other authors.

Since the early 1990s, there has been an increasing interest to apply VSC to TDS. Consequently, that trends has led to a significant number of published papers. For the first time, Luo and De La Sen (1993) gave a satisfactory solution to the stabilization problem of TDS through the use of VSC. Shyu & Jun-Juh (1993), proposed an integral sliding surface design approach which is seen as a kind of constrained VSC. A survey is given on the issue in (Steinberger, *et al.*,2020).There are significant advancements from Richard *et al* (2001), Gouaisbaut and Peaucelle (2006), Gouaisbaut *et al.*(2009) and most importantly from Jafarov (2009). These authors introduced the concept of matrix norms, Linear Matrix Inequalities (LMI) and Riccati Equation. Hung *et al.* (1993), mentioned the fact that the whole research field was yet to be seriously investigated from that perspective. Koshkouei and Zinober (1996), were amongst the pioneers to use VSC for the purpose of stabilization. In this regard, much breakthrough was made in (Jafarov,2011), where the definition of delay dependent and delay independent stability have been clarified and emphasized. These aspects are also well accounted for in (Sename,2013,2014), where the Lyapunov-Krasovskii and Lyapunov-Razhumikin approaches are well described. The review by Keqin *et al*.(2003a) is one of the best on the issue. Kharitonov (1999) ponders on these concepts and approaches as well as on the characteristic equation and its roots. From the same perspective, that is Lyapunov-Krasovskii functional, Emilia Fridman, in (Fridman,2014), addresses time-varying delays. In (Wu *et al.*, 1996), the issue of constructing a sliding surface for linear systems from a classical Lyapunov point of view is addressed. Such a design was successfully applied in (Rimbe *et al.*, 2017) to a VSC for a non-delayed ship. The purpose of this paper is to achieve a similar goal for a delayed ship model. More trends in recent developments of VSS and VSC are highlighted in (Steinberger, *et al.*,2019) and (Steinberger, *et al.*,2020).

3. Preliminaries and Mathematical Tools

This section reviews some of the algebraic tools used in subsequent sections.

Given any rectangular matrix $A = A_{mn}$, (that is, a matrix with m lines and n columns), one can always get the **singular value decomposition (SVD)** of A as in (1). The matrix $A^T = A^T_{nm}$, where the rows and columns of A are interchanged, is the transposed of A.

$$A = UDV^T \quad \dots \quad (1)$$

where U, D, V are matrices with the appropriate dimensions and D is diagonal.

$$D = diag(\sigma_1, \sigma_2, \dots \sigma_r) \quad \dots \quad (2)$$

σ_i (positive) are the singular values; $i = 1,2 \dots r \le \min(m, n)$.

The **spectral decomposition** refers to the eigenvalues, when A is square $(m = n)$:

$$Ax = \lambda x = x\lambda \quad \dots \quad (3)$$

Writing this for each eigenvector $x \in \mathbb{R}^n$, we have $AE = E\Lambda$, where E is the matrix formed by the eigenvectors and $\Lambda = diad(\lambda_1, \lambda_2, \dots \lambda_r)$. Assuming $r = n$:

$$A = E\Lambda E^{-1} \quad \dots \quad (4)$$

If $A = A^T$, then the square matrix $A = A_{nn}$ *is* said to be symmetric. If a matrix P is symmetric, the *n* eigenvalues are real. If all eigenvalues are positive, then the matrix P is positive definite $(P > O)$. The eigenvectors corresponding to distinct eigenvalues of a symmetric matrix are orthogonal. For a symmetric positive definite matrix $P = P^T > O$ and normalized (orthogonal) eigenvectors, where $E = U$ and $U^{-1} = U^T$, equation (5) holds.

$$P = U\Lambda U^T \quad \dots \quad (5)$$

It is only in this case, $P = P^T > O$, that the two decompositions coincide $(D = \Lambda)$.

$$P = UDU^T = E\Lambda E^{-1} \quad \dots \quad (6)$$

3.1. Matrix Norm

For any vector $x(t) \in \mathbb{R}^n$, if $x(t) \neq 0$ and $(x^T Ax)$ >0 , then the square matrix A is said to be positive definite: $A > O$. In other words, $(x^T Ax)$>0 is equivalent to $(A > O)$.

Given two positive definite matrices A and ; it can be shown that $(A + B)$ is also positive definite. If $A > O$ (positive definite) , then $-A < O$ (negative definite).

While $\|x\| = \sqrt{x^T x}$ is the euclidian norm of a vector $x = x\,(t) \in \mathbb{R}^n$, the matrix norm $\|A\|$ of a matrix A is defined in equation (7).

$\|A\|$=$max(\frac{\|Ax\|}{\|x\|=1})$=$\sqrt{\lambda_{max}\,(A^TA)}$. … (7)
$\|\alpha A\| = |\alpha|\|A\|$; for any $\alpha \in \mathbb{R}$ … (8)
$\|AB\| \leq \|A\|\|B\|$ … (9)
For any matrices A, B and for any $x = x\,(t) \in \mathbb{R}^n$, conditions (10), (11), (12) hold.
$\|A + B\| \leq \|A\| + \|B\|$ (Schwarz inequality) … (10)
$\|Ax\| \leq \|A\|\|x\|$ … (11)
$\kappa\,(A)$ = $\|A\|\|A^{-1}\| = \frac{\sigma_{max}}{\sigma_{min}} \geq 1$ … (12)
Here $\sigma =$ singular value of A and $\kappa\,(A)$ is the so called condition number of A.
$\lambda_i(A)$; $i = 1, \dots, n$ are eigenvalues of A ; . i.e. $\det(\lambda_i I - A) = 0$.
If $P = P^T > O$, then (13) holds.
$\|P\| = \lambda_{max}\,(P)$ = the maximum eigenvalue of P. … (13)
Rayleigh's quotient is $R(x)$ in (14) for any vector $x \neq 0$.
$R(x) = \frac{x^TAx}{x^Tx} = \frac{x^TAx}{\|x\|^2}$ … (14)
$R(x)$ = $(\frac{x}{\|x\|})^T A \left(\frac{x}{\|x\|}\right) = v^TAv$; $v = \frac{x}{\|x\|}$
Any vector x that minimizes $R(x)$ is an eigenvector : $Ax = \lambda x,$ and so
$\lambda = R(x) = \frac{x^TAx}{x^Tx} =$ for that eigenvector. We have (15), (16), (17) that hold.
$\lambda_{min}\,(A) \leq R(x) \leq \lambda_{max}\,(A)$ … (15)
$min\,R(x)$=$\lambda_{min}\,(A)$ … (16)
$max\,R(x) = \lambda_{max}\,(A)$ … (17)
If $(P^T = P > O)$, then condition (18) holds.
$0 \leq \lambda_{min}\,(P)\|x(t)\|^2 \leq x^T(t)Px(t) \leq \lambda_{max}\,(P)\|x(t)\|^2$ … (18)
Condition (18) is the so called **Rayleigh's principle** for a positive definite matrix P.

3.2. Equivalent Statements for a Definite Matrix

Given two positive definite matrices A , B ; it can be shown that A + B is also positive definite. If $A > O$ (positive definite) , then $-A < O$ (negative definite).

For any given square matrix $A > O$ of size n, the following statements (i) to (vi), are equivalent.

(i). The n pivots of A are strictly positive (they are reals).
(ii). The n determinants in (19) of the main diagonal of the matrix A are positive.
(iii). The n eigenvalues of A are strictly positive (they are reals).
(iv). For any vector $x = x(t) \in \mathbb{R}^n$, if $x(t) \neq 0$; we have (x^TAx) >0 . This definition, based on the energy (Lyapunov's function) is fundamental in control systems.
(v). $A = R^TR$ where R has its columns linearly independent.
(vi). The Cholesky's decomposition $A = LL^T$ is possible. Here, L is a lower triangular matrix.
More on the issue of matrices to be found in (Guerin *et al.*,2008,2020).

$$|A_1| = [a_{11}];\ |A_2| = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix};\ |A_3| = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix};\ \dots\ ;\ |A_n| = det(A) \qquad \dots \qquad (19)$$

3.3. Schur Complement

Given the block partitioned matrix in (20), the Schur complements are defined next.

$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ … (20)
Assuming $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} > O$ … (21)
The statement in (21) is equivalent to each of the conditions in (22) and (23).
$(A_{11} - A_{12}A_{22}^{-1}A_{21})$ > 0 and $A_{11} > 0$ if $det\,(A_{22}) \neq O$ … (22)
$(A_{22} - A_{21}A_{11}^{-1}A_{12})$ > 0 and $A_{22} > 0$ if $det(A_{11}) \neq O$ … (23)
The terms $(A_{11} - A_{12}A_{22}^{-1}A_{21})$ and $(A_{22} - A_{21}A_{11}^{-1}A_{12})$ are the Schur complements.

4. System Description

In this section, a description of the delayed system is given. A single input system with a single fixed state-delay is considered in (24), (25) and (26).

$$\dot{\boldsymbol{x}}(t) = A_0\boldsymbol{x}(t) + A_1\boldsymbol{x}(t-h) + Bu(t);\ t > 0 \qquad \ldots \qquad (24)$$
$$\boldsymbol{x}(t) = [x_1(t) \quad \ldots \quad x_n(t)]^T ; \qquad \ldots \qquad (25)$$
$$\boldsymbol{x}(t) = \boldsymbol{\varphi}(t),\ \ for\ \ t\,\epsilon\,[-h\,;\ 0]; \qquad \ldots \qquad (26)$$

$0 \le h \le h_{1max}$; where h_{1max} is a value of the delay at which stability is lost.
$\boldsymbol{\varphi}(t)$: $(n \times 1)$ is a known continuous vector valued initial state condition function;
$\boldsymbol{x}(t)$: $(n \times 1)$is the measurable state vector ;
$u(t)$is the control input scalar, in this SISO (Single Input / Single Output) system;
B : $(n \times 1)$; is the known constant input matrix, with full rank ;
h is the constant scalar, always positive, time- delay;
A_0 , A_1 : $(n \times n)$; are the known constant system matrices .

4.1. Delay Independent Lyapunov-Krasovsskii Stability Approach

It is assumed in this section that, the reader is familiar with the classical Lyapunov stability theory for non delayed systems. The TDS described in (24) is said to be infinite dimensional, because its characteristic polynomial $P(s)$ in (27), has an infinite number of roots (Kharitonov,1999). The characteristic polynomial is the denominator of the transfer function of a SISO system.

$$P(s) = \det(sI - A_0 - A_1e^{-sh}) = 0 \qquad \ldots \qquad (27)$$

When using the Lyapunov stability analysis, the infinite dimensional nature of the system in (24) requires a new approach as compared to the classical Lyapunov theory. This idea is conceptualized in (Fridman,2014), (Sename,2013) and (Keqin *et al.*, 2003a, 2003b). It is widely acknowledged that, the credit for the use of a functional instead of an ordinary classical Lyapunov function, goes to Krasovskii (1956). Hence the Lyapunov-Krasovskii approach.
Consider the delayed system (24) and the Lyapunov-Krasovskii functional $V(\,x,t)$ in (28), where matrices P , Q are symmetric positive definite.

$$V(\,x,t) = x^T(t)Px(t) + \int_{-h}^{0} x^T(t+\theta)Qx(t+\theta)d\theta \qquad \ldots \qquad (28)$$

This functional is positive definite and hence the conditions (i) and (ii) are satisfied.

(i). $V(x,t) > 0$ for all $x\,(t) \in \mathbb{R}^n$ and for all $t \in \mathbb{R}$.
(ii). $V(x,t) = 0$ if and only if $x = 0$.

Showing that $\frac{dV}{dt} = \dot{V}(x(t)) < 0$ along the trajectories, is proof that the system is stable.

$$\frac{dV}{dt} = \dot{V}(x(t)) = \dot{x}^T(t)Px(t) + x^T(t)P\dot{x}(t) + x^T(t)Qx(t) - x^T(t-h)Qx^T(t-h) =$$
$$[A_0x(t) + A_1x(t-h)]^T Px(t) + x^T(t)P\big[A_0x(t) + A_1x(t-h)\big] + x^T(t)Qx(t) - x^T(t-h)Qx^T(t-h) = [x^T(t){A_0}^T + x^T(t-h)A_1^T]\, Px(t) + x^T(t)P\big[A_0x(t) + A_1x(t-h)\big] + x^T(t)Qx(t) - x^T(t-h)Qx^T(t-h) = [x^T(t){A_0}^T Px(t) + x^T(t-h)A_1^T\, Px(t)] + x^T(t)PA_0x(t) + x^T(t)PA_1x(t-h) + x^T(t)Qx(t) - x^T(t-h)Qx^T(t-h)$$

$$= \begin{pmatrix} x(t) \\ x(t-h) \end{pmatrix}^T \begin{pmatrix} PA_0 + {A_0}^T P + Q & PA_1 \\ A_1^T P & -Q \end{pmatrix} \begin{pmatrix} x(t) \\ x(t-h) \end{pmatrix} < O$$

The system is asymptotically stable for any delay if there exist positive symmetric matrices $P = P^T > O$ and $Q = Q^T > O$, such that condition (29) is satisfied.

$$\begin{pmatrix} PA_0 + {A_0}^T P + Q & PA_1 \\ A_1^T P & -Q \end{pmatrix} < O \qquad \ldots \qquad (29)$$

The result in (29) can be found in (Sename, 2013,2014), Keqin *et al.(*2003), and is referred to as the delay-independent stability criterion.

5. Variable Structure Controller Design Methodology

The objective is to design a sliding mode controller that stabilizes the system in (24). As recalled by Jafarov (2009), a primary objective in the design of any control system is to maintain stability and performance level in system dynamics and in the working environment to a certain desired degree. Here, delay-independent sufficient conditions will be given for the existence of a sliding mode and the asymptotic stability of the closed-loop system. The design of any VSC is a two stage procedure. A first step is the design of the **switching surface** $s(x) = 0$ to represent a desired system dynamics, which is of lower order than the given plant. The second step is the design of the **variable structure control law** $u(x,t)$ such that any state $\boldsymbol{x}$(t) outside the switching surface is driven to reach the surface in finite time from any arbitrary initial conditions.

5.1. Sliding Surface Design

A sliding surface G isconsideredin (30).

$$G = \{x|\, g = s(x) = Sx(t) = 0\} \quad \ldots \quad (30)$$

This surface G is of dimension $(n-1)$, and S is a sliding surface matrix of full rank.
It is a normal vector to G. It is easily seen that equation (31) holds.

$$S = \boldsymbol{\nabla}^{\boldsymbol{T}} g = S = \boldsymbol{\nabla}^{\boldsymbol{T}} s(\boldsymbol{x}) \quad \ldots \quad (31)$$

$$S = [S_1 \quad S_2 \quad \ldots \quad S_n] \quad \ldots \quad (32)$$

It is shown in (Kwon and Pearson,1977) and in (Feliachi and Thowsen,1981), that stabilization of the system (24) can be achieved by the memoryless feedback $u_f(x,t)$ in (33) where $K = B^T P$ and $P = P^T > O$ is the solution to the Riccati equation in (34).

$$u_f(x,t) = -Kx = -B^T Px \quad \ldots \quad (33)$$

$$A_0 P + PA_0{}^T - PB^T BP + Q = 0 \quad \ldots \quad (34)$$

$Q = Q^T > O$ is to be chosen so that the existence condition of the sliding motion is guaranteed. Based on that result and similarly to what has been done in (Young, *et al*., 1999), (Wu et *al*.,1996), and in (Rimbe, 2005) , (Rimbe*et al.*, 2017), the surface matrix S is selected as in (35).

$$S = K = B^T P \quad \ldots \quad (35)$$

The switching function is defined in (36) and the signum function $sign(g)$ is in (37) and (38).

$$g = s(x,t) = Sx(t) \quad \ldots \quad (36)$$

$$sign(g) = \begin{cases} 1 & if \quad g = s(x,t) > 0 \\ 0 & if \quad g = s(x,t) = 0 \\ -1 & if \quad g = s(x,t) < 0 \end{cases} \quad \ldots \quad (37)$$

$$sign(g) = \frac{Sx}{|Sx|} = \frac{g(t)}{|g(t)|} = \frac{s(t)}{\|s(t)\|} if \ s(x,t) \neq 0 \quad \ldots \quad (38)$$

$$sign(g) = 0 \ if \ \ s(x,t) = 0$$

5.2. The Control Law Design

As in (Jafarov,2009), the following type of VSC law in (39) can be considered to prove <u>Theorem 1</u> and <u>Lemma 1.</u>

$$u(\boldsymbol{x},t) = -[k\|x\|](B^T PB)^{-1} \frac{s(t)}{\|s(t)\|} \quad \ldots \quad (39)$$

$k > 0$ is a design parameter and $u(\boldsymbol{x},t)$ is a control vector.

However, from (Drazenovic, 1969), (Zinober, 1981) and (Rimbe, 2005), (Rimbe *et al.*,2017), it is sufficient, for a certain regulator without external disturbances, to use the simplified control law in (40).

$$u(x,t) = -k|x_1| \frac{s(t)}{\|s(t)\|} \quad \ldots \quad (40)$$

Notice that $|x_1| \leq \|x\|$, and so the conditions of <u>Lemma 1</u> and <u>Theorem 1</u> are fulfilled.

5.3. Sliding Mode Existence Conditions

We have selected the sliding surface matrix in (31), (32), (35) and defined the controller (39). The next step is now to choose the design parameter k such that on the sliding surface, a sliding mode and the closed-loop system are globally asymptotically stable.

Lemma 1: (Jafarov,2009) :

Given the time-delay system (24) driven by the controller (39), a stable sliding mode can always be generated on the surface (31),(30), (32) and (35) if the following conditions (41), (42) and (43) are satisfied.

There exist a matrix $Q_s > O$ and a scalar $k > 0$ such that:

$$P_sA_0 + A_0^TP_s + \beta_vP_s - 2k\omega < O \ ; \qquad \ldots \qquad (41)$$

Where $P_s = PBB^TP = {P_s}^T \geq 0$;

$S = B^TP$; $\beta_v > 0$; $\omega = \frac{1}{\sqrt{\lambda_{\max}(P_s)}}$ is a given constant.

$$P_sA_0 + {A_0}^TP_s + \beta_vP_s - 2k\omega = -Q_s < O;$$

$$P_sA_0 + A_0^TP_s + \beta_vP_s - 2k\omega + Q_s = O. \qquad \ldots \qquad (42)$$

$$H_s = \begin{pmatrix} P_sA_0 + {A_0}^TP_s + \beta_vP_s & P_sA_1 \\ A_1^TP_s & -\beta_vP_s \end{pmatrix} < O \qquad \ldots \qquad (43)$$

Proof:

A Lyapunov-Krasovskii functional is considered as $V(s,t) \geq 0$:

$V(s,t) = s^T(t)s(t) + \beta_v \int_{t-h}^{t} s^T(\theta)s(\theta)d\theta$. $\beta_v > 0$.

$s(t) = B^TPx = Sx$and so $\dot{s}(t) = S\dot{x} = B^TP\dot{x}$; $P_s = PBB^TP \geq O$.

$s^T(t)s(t) = x^T(t)(PBB^TP)x(t) = x^T(t)P_sx(t) = \|s(t)\|^2 = \lambda_{\max}(P_s)\|x(t)\|^2$.

$s^T(t-h)s(t-h) = x^T(t-h)P_sx(t-h) = \lambda_{\max}(P_s)\|x(t-h)\|^2$.

$\dot{V}(s,t) = \frac{dV}{dt} = \dot{s}^T(t)s(t) + s^T(t)\dot{s}(t) + \beta_v[s^T(t)s(t) - s^T(t-h)s(t-h)] = 2s^T(B^TP)\dot{x} + \beta_v[s^Ts - s^T(t-$

$hst-h=$

$2s^T(B^TP)[A_0x(t) + A_1x(t-h) + Bu(x,t)]\beta_v[x^T(t)P_sx(t) - x^T(t-h)P_sx(t-h)]$=$x^T(t)[P_sA_0 + A_0P_s +$

$\beta vPsxt-2sTtBTPBBTPB-1kxt+2xTt(PsA1)x(t-h)-\beta vxTt-h(Ps)x(t-h)$

Note that $[B^TPB(B^TPB)^{-1}] = I$. In general, $\lambda_{\min}(P_s) = 0$.

Then follows

$-2s^T(t)[B^TPB(B^TPB)^{-1}]k\|x(t)\| \leq -2k\|x(t)\|\|s(t)\|$.Using Schwarz inequality :

$\|s(t)\| = \|B^TP\boldsymbol{x}(t)\| \leq \|B^TP\|\|x(t)\|$=$\sqrt{\lambda_{\max}(P_s)}\|x(t)\|$

$\|x(t)\|\|s(t)\| \geq \omega\|s(t)\|\|s(t)\| = \omega\|s(t)\|^2 = \omega s^T(t)s(t)$

$\omega = \frac{1}{\sqrt{\lambda_{\max}(P_s)}}$ and $\|x(t)\| \geq \frac{1}{\sqrt{\lambda_{\max}(P_s)}}\|s(t)\|$;

$-2k\|x(t)\|\|s(t)\| \leq -2k\omega s^T(t)s(t) = 2kx^T(t)PBB^TPx(t) = 2kx^T(t)P_sx(t)$.Finally, as in (Jafarov 2009),

$\dot{V}(s,t) \leq x^T(t)\left[P_sA_0 + {A_0}^TP_s + \beta_vP_s\right]x(t) - 2k\omega x^T(t)P_sx(t) + 2x^T(t)(P_sA_1)x(t-h) + \beta_v[x^T(t-h)P_sx(t-h)]$
$=$

$$\begin{pmatrix} x(t) \\ x(t-h) \end{pmatrix}^T \begin{pmatrix} P_sA_0 + {A_0}^TP_s + \beta_vP_s - 2k\omega P_s & P_sA_1 \\ A_1^TP_s & -\beta_vP_s \end{pmatrix} \begin{pmatrix} x(t) \\ x(t-h) \end{pmatrix} < 0$$

$\dot{V} \leq z^T(t)H_sz(t) < O$ where $z(t) = [x(t) \quad x(t-h)]^T$

$$\dot{V} < O => H_s = \begin{pmatrix} P_sA_0 + {A_0}^TP_s + \beta_vP_s - 2k\omega P_s & P_sA_1 \\ A_1^TP_s & -\beta_vP_s \end{pmatrix} < O$$

It is concluded that a stable sliding mode is generated on the switching surface $s(t)$=0.

5.4. Stability of the closed-loop system

Theorem 1. (Jafarov,2009).

Suppose that the conditions of Lemma 1 hold. Then the time-delay system (24), driven by the sliding mode controller (39) on the sliding surface in (30),(32) and (35) is globally asymptotically stable, if the following conditions (44) and (45) are satisfied:

$$PA_0 + A_0{}^T P + R_1 - 2k\omega\lambda_{min}\,[(B^T PB)^{-1}](PBB^T P) \quad = -Q_1 < O \qquad \ldots \qquad (44)$$

$$R_1 = R_1^T > O\ ;$$

$$H_1 = \begin{pmatrix} -Q_1 & PA_1 \\ A_1^T P & -R_1 \end{pmatrix} < O\ ; \qquad \ldots \qquad (45)$$

Proof:

Choose the Lyapunov-Krasovskii functional candidate as $V(x,t) \geq 0$:

$V(x,t) = x^T(t)Px(t) + \beta_v \int_{t-h}^{t} x^T(\theta)R_1 x(\theta)d\theta \quad ;\beta_v > 0$, is a positive given constant.

$$\dot{V}(x,t) = \frac{dV}{dt} = x^T(t)\left(PA_0 + A_0{}^T P\right)x(t) + 2x^T(t)PA_1x(t-h) + \beta_v[x^T(t)R_1x(t) - x^T(t-h)R_1x(t-h)]-$$

$$2\|x(t)\|x^T PB(B^T PB)^{-1}k\frac{s(t)}{\|s(t)\|} = -2k\|x(t)\|\frac{s^T\left(t\left(B^T PB\right)^{-1}\right)s(t)}{\|s(t)\|} \leq -2k\lambda_{min}\,[(B^T PB)^{-1}]\|x(t)\|\frac{\|s(t)\|^2}{\|s(t)\|} =$$

$$-2k\lambda_{min}\,[(B^T PB)^{-1}]\|x(t)\|\|s(t)\|.$$

$$s^T = x^T(t)PB\ ; \quad R_1 = R_1^T > O$$

$$\dot{V}(x,t) \leq x^T(t)\left[PA_0 + A_0{}^T P + R_1 - 2k\omega\lambda_{min}\,[(B^T PB)^{-1}]\,PBB^T P\right]x(t) + 2x^T(t)(PA_1)x(t-h)$$
$$+ [x^T(t-h)R_1x(t-h)]$$

$$\dot{V}(x,t) \leq \begin{pmatrix} x(t) \\ x(t-h) \end{pmatrix}^T \begin{pmatrix} -Q_1 & PA_1 \\ A_1^T P & -R_1 \end{pmatrix} \begin{pmatrix} x(t) \\ x(t-h) \end{pmatrix} < O$$

which implies (44) and (45).

Remarks

If $\beta_v = 1$ and $2k\omega\lambda_{min}\,[(B^T PB)^{-1}]$=1 then condition (ii) of Theorem 1 reduces to

the standard Riccati equation in (34), that is : $A_0P + PA_0{}^T - PB^T BP + Q = 0$; where $Q = R_1 + Q_1 \geq O$.

Then k can be found as in (46) below.

$$k = 1/(2\omega\lambda_{min}\,[(B^T PB)^{-1}]) \qquad \ldots \qquad (46)$$

5.5. Design Procedure

The following steps are for practical application of the theorem and the control design.

1. Solve Riccati Equation $A_0P + PA_0{}^T - PB^T BP + Q = 0$

2. Compute $P_s = PBB^T P$

3. Compute $\omega = \frac{1}{\sqrt{\lambda_{max}\,(P_s)}}$ ***;***

4 .Compute $\lambda_{min} = \lambda_{min}\,[(B^T PB)^{-1}]$

5. Find $k = 1/(2\omega\lambda_{min})$

6. Check whether M_1 is negative definite .

$$M_1 = PA_0 + A_0{}^T P + R_1 - 2k\omega\lambda_{min}\,[(B^T PB)^{-1}]P_s$$

7. Check whether H_1 is negative definite : $H_1 = \begin{pmatrix} -Q_1 & PA_1 \\ A_1^T P & -R_1 \end{pmatrix}$

If $M_1 < O$ and $H_1 < O$, then we can design a stabilizing VSC for the system.

8. Sliding Surface and Switching function

$S = B^T P$ and $g = s(x,t) = Sx(t)$

9. Variable Structure Controller (VSC)

$u(\boldsymbol{x},t) = -k|x_1|sign(g)$

5.6. Equation of the Closed-loop System

The system in (24) with $n = 3$ is recalled below:

$\dot{x}(t) = A_0 x(t) + A_1 x(t-h) + Bu(x,t)$;
Or equivalently:

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 \\
\dot{x}_3 &= -\alpha_0 x_1(t) - \alpha_1 x_2(t) - \alpha_3 x_2(t) - \beta_0 x_1(t-h) - \beta_1 x_2(t-h) - \beta_2 x_3(t-h) + u(x,t)
\end{aligned}$$

In sliding mode , we have:
$Sx = S_1 x_1 + S_2 x_2 + S_3 x_3 = 0 \quad => x_3 = -\frac{S_1}{S_3} x_1 - \frac{S_2}{S_3} x_2$
And so, the equations (47) describe the closed-loop system in sliding mode.

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\frac{S_1}{S_3} x_1 - \frac{S_2}{S_3} x_2
\end{aligned} \qquad \ldots \qquad (47)$$

6. Results

$$\begin{aligned}
&\dot{x}(t) = A_0 x(t) + A_1 x(t-h) + B\, u(x,t) \\
&x(t) = \phi(t) \quad if \quad t \epsilon[-h;\ 0] \ , \ h > 0 . \\
&x(t) = [x_1 \quad x_2 \quad x_3]^T \ ; t > 0.
\end{aligned} \qquad \ldots \qquad (48)$$

$$A_0 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\alpha_0 & -\alpha_1 & -\alpha_2 \end{bmatrix} ;\ A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\beta_0 & -\beta_1 & -\beta_2 \end{bmatrix} ;\ B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$Q = 2(\beta_0^2 + \beta_1^2 + \beta_2^2) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\alpha_0 = 0.3240 \ ; \ \alpha_1 = 1.8000; \ \alpha_2 = 2.7000; \qquad \ldots \qquad (49)$$

$\beta_0 = \alpha_0/2$; ;$\beta_1 = \alpha_1/2\beta_2 = \alpha_2/2$;

$x(t) = \phi(t) = [10 \quad 0 \quad -5]^T \ if \quad t \epsilon[-h;\ 0]$

$u(\boldsymbol{x}, t) = -k|x_1| sign(g); \quad g(t) = Sx(t);$

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 \\
\dot{x}_3 &= -\alpha_0 x_1(t) - \alpha_1 x_2(t) - \alpha_3 x_2(t) - \beta_0 x_1(t-h) - \beta_1 x_2(t-h) - \beta_2 x_3(t-h) + u(x,t)
\end{aligned}$$

We have the following results:

$$P = \begin{pmatrix} 11.0921 & 8.7530 & 2.1497 \\ 8.7530 & 15.5030 & 3.9614 \\ 2.1497 & 3.9614 & 2.5311 \end{pmatrix}; \quad S = [2.1497 \quad 3.9614 \quad 2.5311] \text{ and } k = 2.5846.$$

We have the reduced order closed loop system in (50).

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -0.8493 x_1 - 1.5651 x_2
\end{aligned} \qquad \ldots \qquad (50)$$

For this system, stable sliding mode is theoretically guaranteed for any delay h, provided the conditions of Lemma 1 and Theorem 1 are met.

The following figures (Fig.1) to (Fig.14) are the different plots of the states, the control law and the switching function as well. A time response of the system consists of two phases: hitting or reaching mode and the sliding mode. These two modes clearly appear on the plots. During sliding mode, the states remain on the surface $s(t) = Sx(t) = 0$.

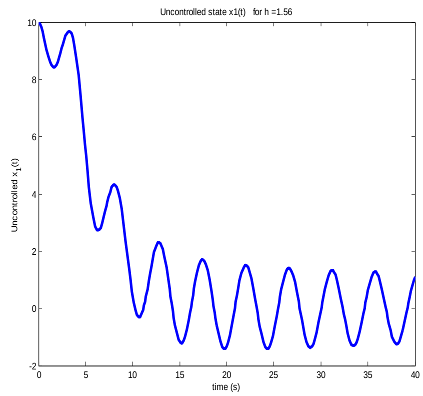

Fig.1- Uncontrolled state $x_1(t)$ for $h = 1.56$

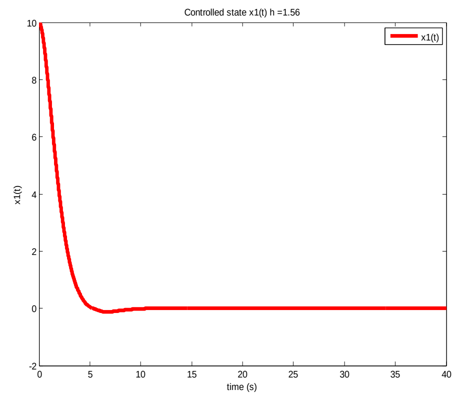

Fig.2- Controlled state $x_1(t)$ for $h = 1.56$

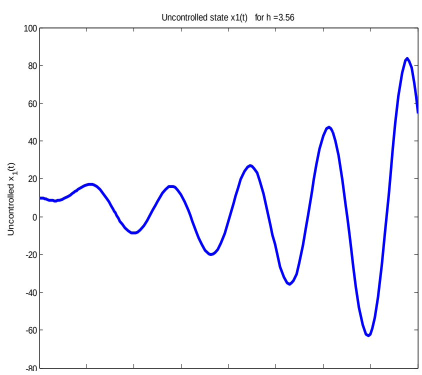

Fig.3- Uncontrolled state $x_1(t)$ for $h = 3.56$

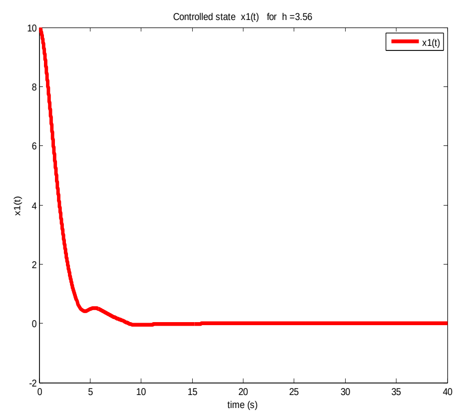

Fig. 4- Controlled state $x_1(t)$ for $h = 3.56$

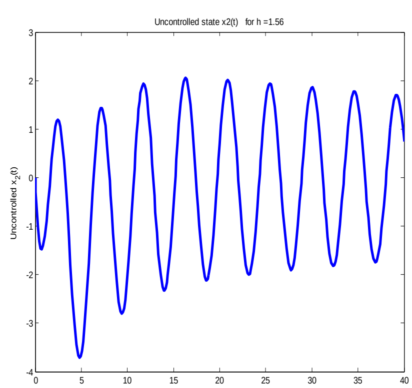

Fig.5-Uncontrolled state $x_2(t)$ for $h = 1.56$

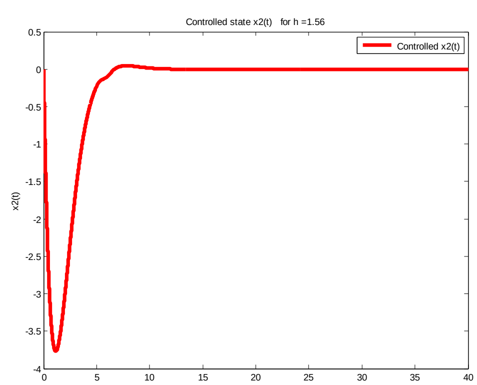

Fig.6-Controlled state $x_2(t)$ for $h = 1.56$

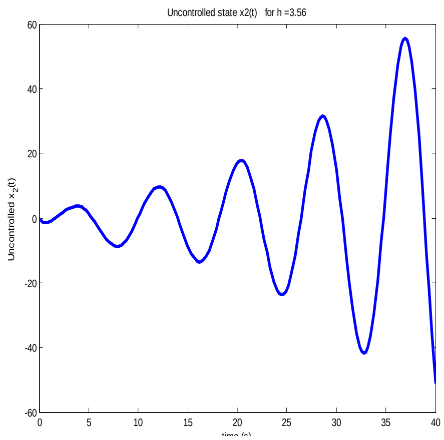

Fig.7-Uncontrolled state $x_2(t)$ for $h = 3.56$

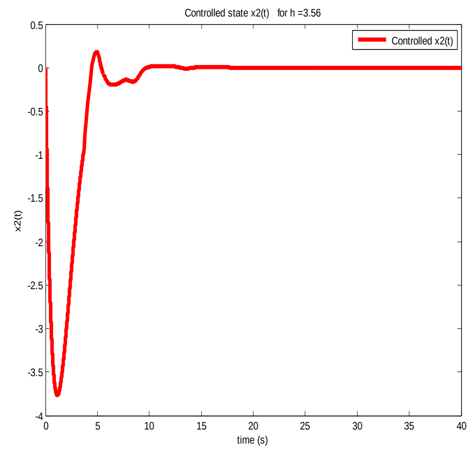

Fig. 8-Controlled state $x_2(t)$ for $h = 3.56$

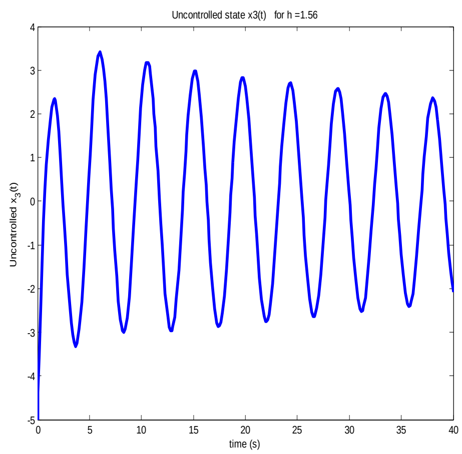

Fig.9-Uncontrolled state $x_3(t)$ for $h = 1.56$

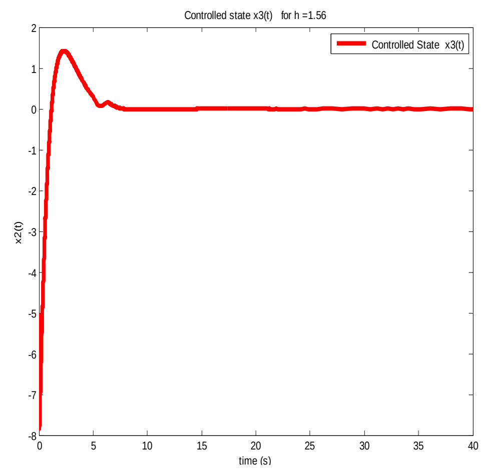

Fig.10-Controlled state $x_3(t)$ for $h = 1.56$

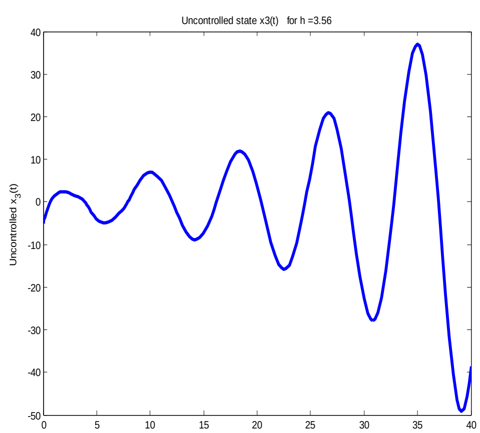

Fig.11-Uncontrolled state $x_3(t)$ for $h = 3.56$

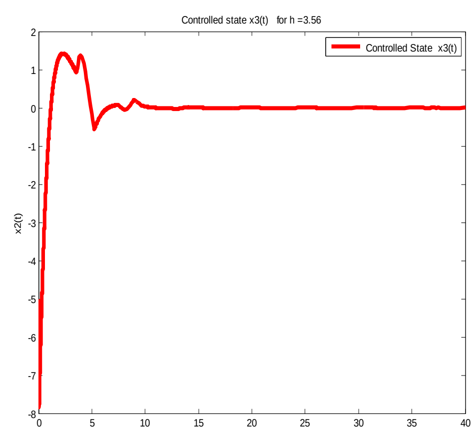

Fig.12-Controlled state $x_3(t)$ for $h = 3.56$

Below are the plots of the control law and the switching function for $h = 1.56$

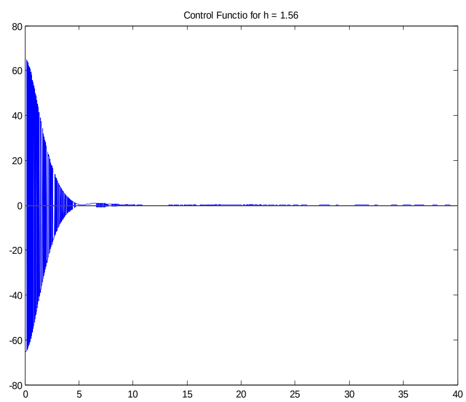

Fig.13-Control law for $h = 1.56$

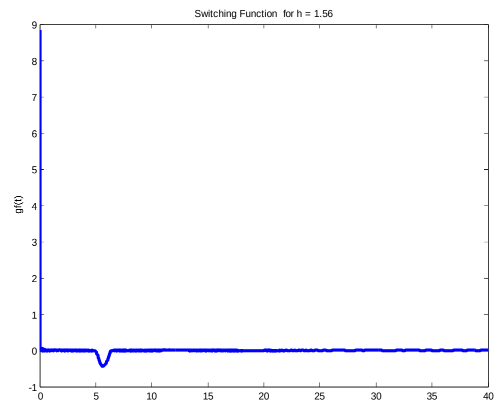

Fig.14-Switching function for $h = 1.56$

7. Conclusion

This paper has presented simulation results for a variable structure control of a time delay system. A single state delay dependent third order system, is stabilized by this method, in spite of disastrous effects of the delay. It has been shown that, provided a system has a stabilizing feedback with known Lyapunov-Krasovskii functional, a sliding surface can be obtained. Sufficient conditions for sliding motion on such a sliding surface and asymptotic stability of the closed loop system are guaranteed, regardless of the initial conditions.

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