Generalization of an Euler Theorem

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Abstract. *In this note, Euler's Theorem is generalized using mathematical induction.*

1. Introduction

The following result, known as the Euler's Theorem, is established in the courses of modern geometry.

"If A, B, C and D are any four points in a line, then the directed segments AB, CD, AC, DB, AD and BC satisfy the relationship
$AB \cdot CD + AC \cdot DB + AD \cdot BC = 0$"
In this note we start with a classic proof of the Euler's Theorem, and then generalize it by mathematical induction on the number of points.

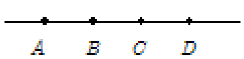
2. An Euler Theorem

Euler's Theorem. *If A, B, C and D are any four points in a line, then the segments*

AB, CD, AC, DB, AD and BC satisfy the relationship

$AB \cdot CD + AC \cdot DB + AD \cdot BC = 0.$

Proof Consider the points A, B, C and D on a line, and let AB, CD, AC, BD, AD and BC be the directed segments determined by them, as shown below.



Hence

$$AB = AD - BD = DB - DA$$
$$AC = AD - CD = DC - DA$$
$$BC = BD - CD = DC - DB;$$

from here we have

$$\begin{aligned}
&AB \cdot CD + AC \cdot DB + AD \cdot BC = \\
&= (DB - DA) \cdot CD + (DC - DA) \cdot DB + (DC - DB) \cdot AD \\
&= DB \cdot CD - DA \cdot CD + DC \cdot DB - DA \cdot DB + DC \cdot AD - DB \cdot AD \\
&= DB \cdot CD - DA \cdot CD - DB \cdot CD + AD \cdot DB + DA \cdot CD - AD \cdot DB \\
&= 0.
\end{aligned}$$

3 Generalization of the Euler's Theorem

The Euler Theorem of the previous section is generalized as follows:

Proposition 1 *If $A_1, A_2, \ldots, A_n$ are any $n \geq 4$ points in a line, then the directed segments*

$A_1A_2,\ A_3A_n,\ A_1A_n,\ A_2A_{n-1},\ A_1A_i,\ A_{i+1}A_{i-1}$

for $3 \leq i \leq n-1$, satisfy:

$$A_1A_2 \cdot A_3A_n + \sum_{i=3}^{n-1} A_1A_i \cdot A_{i+1}A_{i-1} + A_1A_n \cdot A_2A_{n-1} = 0.$$

Proof. If $n = 4$, the theorem is the Euler's Theorem. Now, we suppose the result is true for $n = k > 4$, that is

$$A_1A_2 \cdot A_3A_k + \sum_{i=3}^{k-1} A_1A_i \cdot A_{i+1}A_{i-1} + A_1A_k \cdot A_2A_{k-1} = 0.$$

So, for $n = k + 1$, we have

$$\begin{aligned}
&A_1A_2 \cdot A_3A_{k+1} + \sum_{i=3}^{k} A_1A_i \cdot A_{i+1}A_{i-1} + A_1A_{k+1} \cdot A_2A_k = \\
&= A_1A_2 \cdot A_3A_{k+1} + \sum_{i=3}^{k-1} A_1A_i \cdot A_{i+1}A_{i-1} + \\
&\qquad + A_1A_k \cdot A_{k+1}A_{k-1} + A_1A_{k+1} \cdot A_2A_k \\
&= A_1A_2 \cdot A_3A_{k+1} - A_1A_2 \cdot A_3A_k - A_1A_k \cdot A_2A_{k-1} + \\
&\qquad + A_1A_k \cdot A_{k+1}A_{k-1} + A_1 \cdot A_{k+1} \cdot A_2A_k \\
&= A_1A_2 \cdot (A_3A_{k+1} - A_3A_k) + A_1A_k \cdot (-A_2A_{k-1} + A_{k+1}A_{k-1}) + \\
&\qquad + A_1A_{k+1} \cdot A_2A_k \\
&= A_1A_2 \cdot A_kA_{k+1} + A_1A_k \cdot A_{k+1}A_2 + A_1A_{k+1} \cdot A_2A_k \\
&= 0.
\end{aligned}$$

The last equality is obtained by applying the Euler's Theorem to the points A_1, A_2, A_k y A_{k+1}.

References

Shively, Levi S., *Introducción a la Geometría Moderna*, CECSA, México, 1982.

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