Elastic Buckling Loads of Hinged Frames by the Newmark Method

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Abstract

A numerical method for the solution of the elastic stability of hinged frames is presented. The Newmark iterative procedure to perform elastic buckling analyses for isolated columns is extended for use in computing buckling loads and buckling modes in frames with hinged columns. The method is illustrated in details by means of different cases of single-storey portal frames commonly used in commercial buildings. Comparisons of obtained results with other well known methods show very good agreement.

Keywords: Numerical Analysis; Stability; Buckling; Frames.

Introduction

The storey of instability problem is a unique one with continuity over more than two hundred years. The Euler formula for the elastic critical buckling load of a slender column is one of the earliest engineering design formulas that are in use today. Euler proved theoretically that there is another criterion for column strength which is independent of crushing or yielding of material. This is called the "Stability Limit Load" of a column, and is given by $P_E = \pi^2 EI/L^2$ in which L is the column length, E is the elastic modulus of the material and I is the moment of inertia of the column cross section. This formula is applicable to all elastic columns provided that the right hand side is multiplied by an appropriate constant coefficient corresponding to the end conditions. The natural growth of design methods has led to compare the behavior of columns in structures with isolated ones. In practice most of the frameworks have some degree of rigidity in the connections, consequently, when local loads induce flexure in one of the members of a frame, joints at the end of the member rotate and the flexural disturbance spreads to the adjacent members.

The analysis of such flexural effects in rigid-jointed frameworks has been carried out manually for many years by methods such as the moment distribution and slope deflection, before the introduction of numerically based solutions relying on modern computer capabilities. The methods of analysis are usually linear, i.e. stresses and displacements are found to be directly proportional to the load. If a more general analysis is made by taking account the reduction in the flexural stiffness of a strut due to the presence of compressive forces, stresses and displacements deviate more and more from the simple linear solution as the load increases. Flexural stresses and displacements become very large as the load approaches some critical value. This is the "Elastic Critical Load of the Framework", and it is analogous with the Euler load for a slender elastic column. The critical load of a bar with uniform or non-uniform cross section can be calculated by a numerical method of double integration (Newmark 1943). Instead of assuming the deflection y as some function of x, the beam is divided into segments and a numerical value of deflection is assumed at each division point along the beam.

The subsequent calculations are made, determining ordinates to the *M/EI* diagram and new values of deflections at each site. If these are equal to the assumed deflections at every division point, then the required critical load P and the buckling mode are determined. If they are not equal, the new set of deflections is assumed and the calculations are repeated. This procedure is successful because the results of each cycle yield better deflections and the procedure converges to the exact buckling mode after a few numbers of cycles of iteration. In this paper, the technique of Newmark's numerical method (Newmark 1943), applied to columns, is extended for use in computing buckling loads and buckling modes of hinged frames. The philosophy of the method described herein can be summarized as follows: the buckling load of the structure is the load just enough to maintain it in an assumed buckling configuration. The method involves cycles of iteration in which a new configuration better than the assumed one is obtained at the end of each cycle. The calculations can be repeated until the required degree of accuracy is obtained. In most cases, accurate results are obtained after only few cycles.

Symmetrical Buckling Modes of Two Hinged Frames

Consider the two hinged frame shown in Fig. 1, the end forces and rotations for each member are separately shown in Fig. 2.

The column AB is subjected to three forces at B, namely: vertical force P, end couple X_1, and horizontal force X_2. The end rotation φ at B or C can be expressed in terms of the dimensions and properties of the horizontal beam BC: $\varphi = (L_b/2EI_b)\, X_1$, neglecting the effect of axial force X_2. Our goal is to determine P, X_1 and X_2 (X_2 is directly obtained from X_1) which maintain the column in its assumed buckling shape (y_a) and satisfy the end conditions at B. These end conditions are: (1) rotation φ at B of both column BA and beam BC is equal to $(L_b/2EI_b)\, X_1$, (2) horizontal displacement at B equal zero. Figure 3(a) shows the column AB and the end forces at B. The deformation of the column can be regarded as the superposition of Fig. 3(b) and Fig. 3(c) multiplied by X_1.

On the basis of the assumed values of the chosen deflection shape y_a, the pin ended column AB under only the axial force P is considered, Fig. 3(b), and the resulting deflection shape y_p and rotation at B, φ_p, are calculated. Again, the pin ended column AB is subjected to a unit couple (X_1=1) at end B, Fig. 3(c) and the resulting deflection, $y_{x_1=1}$, and rotation at B, $\varphi_{x_1=1}$, are also determined. Then, the value of the couple X_1 can be obtained from the relation

$$\varphi_p + X_1\varphi_{x_1=1} = \frac{L_b}{2EI_b} X_1 \tag{1}$$

And the new deflection of the column (y) is calculated directly from the following equation

$$y = y_p + X_1 y_{x_1=1} \tag{2}$$

Finally the ratios of assumed deflections y_a to the resulting values y are determined and the critical load suggested by this first cycle is obtained. The results can be obtained by repeating the cycle of calculations. An illustration of the method is given by the following example. The critical load and the corresponding buckling mode are to be determined for the symmetrical two hinged frame of Fig. 1. The solution is found in Fig. 4 where the pin ended column AB is divided into seven sections (6 segments), each segment of length equal to λ, where $\lambda = L_c / 6$.

Line 1 assumed set of deflections y_a representing a sine curve of amplitude 1000, which is the assumed deflection at the middle of the column

Line 2 to 6 corresponding values of moments M_p, angle changes α, equivalent concentrated elastic load $\bar{\alpha}$, average slopes φ_{av}, and deflections y_p are recorded, respectively.

Line 7 the slope at B, due to the axial load P is thus found to be $\varphi_p = (1825+89)P\lambda / EI_c = 1914P\lambda / EI_c$.

The pin ended column AB is then subjected to a unit couple, $X_1 = 1$, at its end B. Normal calculations are shown from line 8 to 10. In line 11 are given the values of average slope φ_{av} in the different segments after assuming a value of 6 in the first segment. In line 12 trial deflections are obtained starting with zero value at A. The resulting trial deflection at B has a value of $1\lambda^2 / 6EI_c$ instead of zero.

Line 13 correction deflections y_c are applied with values varying linearly over the length of the column from $-1\lambda^2 / 6EI_c$ at B to zero at the left end.

Line 14 true deflection, $y_{x_1=1}$ are obtained, line 14 = line 12 + line13

Line 15 true value of the average slope between section 6 and 7 is equal to $(0-9.17)\lambda / EI_c$

Line 16 slope at B due to a unit couple, $\varphi_{x_1=1}$ is equal to $(-9.17-2.83)\lambda / 6EI_c$ or $-12\lambda / 6EI_c$

From Eq. (1), $1914P\lambda / EI_c + X_1(-12\lambda / 6EI_c) = (L_b / 2EI_b)X_1$, where $\lambda = L_c / 6$. By rearranging we get X_1 = (1914 P)/(2 + 3 K_c/K_b) in which $K_c = EI_c / L_c$ and $K_b = EI_b / L_b$. Hence, for $K_c / K_b = 1$, $X_1 = 382.8P$.

Line 17 new deflections y calculated from Eq. (2). Thus, line 17 = line 6 + (382.2) line 14, for example y (at the middle section) = $-3650P\lambda^2 / EI_c + 382.2P\left(13.5\lambda^2 / EI_c\right) = -2789P\lambda^2 / EI_c$.

line 18 gives the ratio y_a / y at every division point. This ratio when equal to unity gives an estimate of the critical load, which appears to be not yet the same at all the division points ranging from 0.34411 to $0.40323EI_c / P\lambda^2$ giving a critical load between 12.39 and $14.52EI_c / L_c^2$.

The critical load calculated from the better ratio $\sum y_a / \sum y$ is $13.078 EI_c / L_c^2$, which is about one and half percent greater than the exact value of $12.88 EI_c / L_c^2$ (Horne and Merchant 1965). Figure 5 shows a complete second cycle. The assumed buckling configuration y_a is obtained by reducing the deflections of line 17 of the previous cycle in the ratio of 1000/2789. It is noticed that the ratio y_a / y now ranges from 0.35254 to $0.36929 EI_c / P\lambda^2$ giving a critical load between 12.69 and $13.29 EI_c / L_c^2$. The critical load calculated from the better ratio $\sum y_a / \sum y$ is $12.94 EI_c / L_c^2$ with a difference of 0.47%. This result demonstrates the rapid convergence of the procedure.

Antisymmetrical mode of buckling of hinged frames

Consider the simple two hinged frame shown in Fig. 6 with antisymmetrical mode of buckling. The end forces and rotation for each member are separately shown in Fig. 7. The column AB is subjected to only two forces at B, namely: vertical force P and a couple X_1. The end rotation φ at B or C can be easily obtained from the horizontal beam: $\varphi = (L_b/2EI_b)\, X_1$. Our goal is to determine P and X_1 which can maintain the column in its assumed buckling shape y_a and at the same time produce an angle of rotation at B equal to ($L_b/6EI_b$). Fig. 8 shows the column AB with an arbitrary sidesway Δ of 1000 units at B, hence $X_1 = 1000\,P$. A complete numerical solution of the considered frame is illustrated in Fig. 9. Details of different steps are given as follows.

First Cycle

Line 1 assumed set of deflection y_a representing a sine curve of amplitude 1000 at the right end.

Line 2 equivalent concentrated elastic loads $\bar{\alpha}$

Line 3 the desired value of slope at B is equal to ($L_b/6EI_b$) X_1. Therefore $\varphi = (k_c / k_b) 1000 P\lambda / EI_c$ and for $k_c / k_b = 1$, $\varphi = 1000 P\lambda / EI_c$. This value is entered at the right end of line 3 between brackets and hence, line 3 is calculated from right to left.

Line 4 new deflection y

Line 5 line 1/line 4

Two subsequent cycles are also completely recorded, and the final one gives a critical load with a value of $1.8216 EI_c / L_c^2$ which is equal the exact value of $1.8213 EI_c / L_c^2$ (Timoshenko and Gere 1961). The suggested starting buckling mode of the fourth cycle (last line) is almost identical to the third one, thus the corresponding shape of the column at the critical condition is also determined with a high degree of accuracy.

Conclusions

The Newmark's double integration procedure is extended for use in computing critical loads and buckling modes of hinged frames. Results obtained show very good agreement with well-known methods. The elastic line of the mode of buckling is determined as a major part of the solution, which gives a clear insight of the behavior of the structure. The method presented here can be used to study buckling of frames with varying cross sections.

References

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Timoshenko, S. P., & Gere, J. M., (1961). "Theory of elastic stability," McGraw-Hill, New York, 2nd ed.

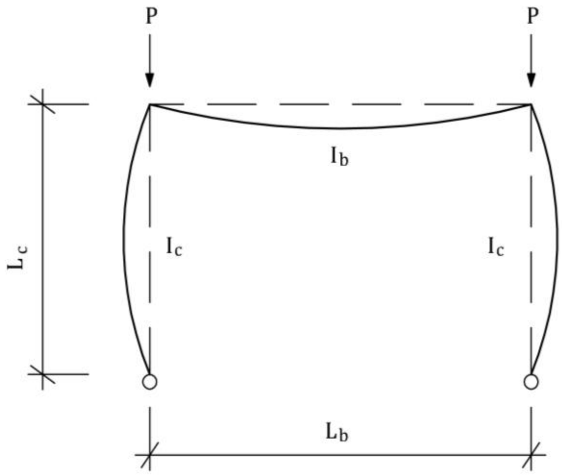


Fig. 1 - Symmetrical two hinged frame (No Side Sway)

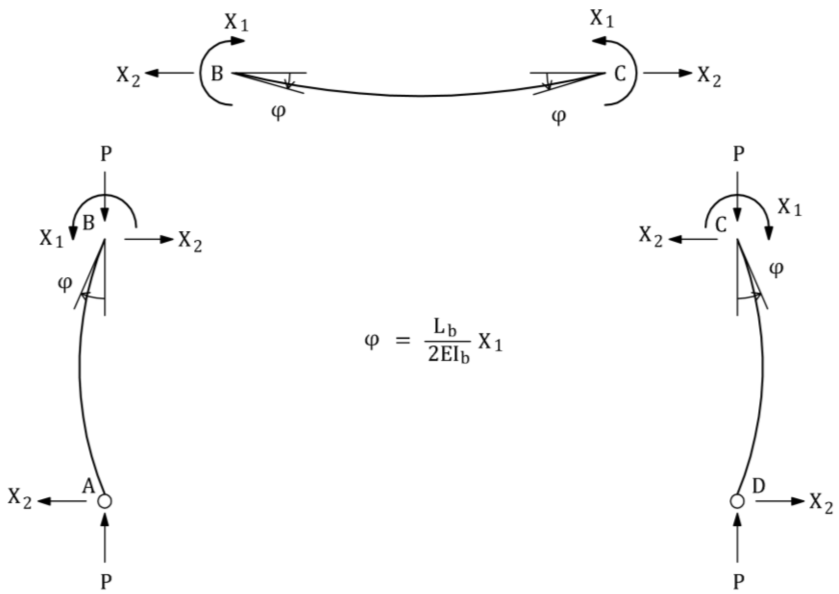


Fig. 2 - Symmetric two hinged frame: end forces and rotations

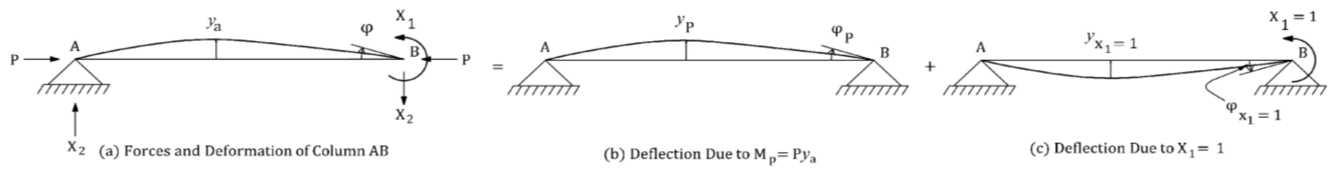
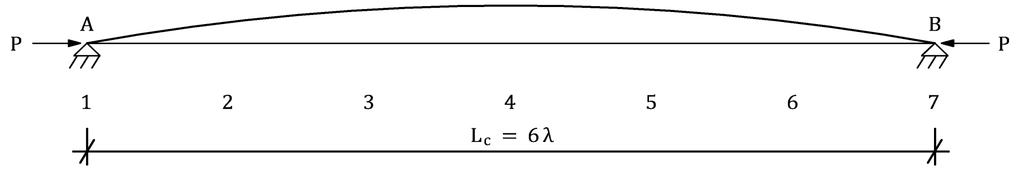
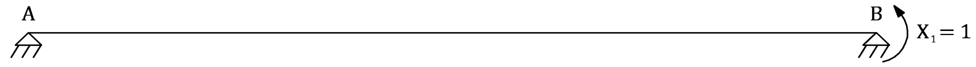


Fig. 3 - Symmetric two hinged frame: superposition, Fig. 3(a) = Fig. 3(b) + X_1 Fig. 3(c)

Cycle Number (1)



	1		2		3		4		5		6		7	
1- y_a	0		-500		-866		-1000		-866		-500		0	
2- Mp	0		-500		-866		-1000		-866		-500		0	P
3- α	0		500		866		1000		866		500		0	P/EIc
4- $\bar{\alpha}$	89		489		847		978		847		489		89	$P\lambda$/EIc
5- φ_{av}		-1825		-1336		-489		489		1336		1825		$P\lambda$/EIc
6- y_p	0		-1825		-3161		-3650		-3161		-1825		0	$P\lambda^2$/EIc
7- φ_p													1914	$P\lambda$/EIc



8- M	0		1		2		3		4		5		6	1/6
9- α	0		-1		-2		-3		-4		-5		-6	1/6EIc
10- $\bar{\alpha}$	-0.16		-1		-2		-3		-4		-5		-2.83	λ/6EIc
11- Ass. φ avg		(6)		5		3		0		-4		-9		λ/6EIc
12- Trial y	0		6		11		14		14		10		1	λ^2/6EIc
13- y_c	0		-0.17		-0.33		-0.5		-0.67		-0.83		-1	λ^2/6EIc
14- $y_{X_1=1}$	0		5.83		10.67		13.5		13.33		9.17		0	λ^2/6EIc
15- $\varphi_{X_1=1}$ avg												-9.17		λ/6EIc
16- $\varphi_{X_1=1}$													-12	λ/6EIc
17- y	0		-1453		-2480		-2789		-2311		-1240		0	$P\lambda^2$/6EIc
18- y_a/y			0.34411		0.34920		0.35855		0.37473		0.40323			EIc/$P\lambda^2$

$$\text{Better Ratio} = \Sigma y_a / \Sigma y = (3732/10273)\ \text{EIc}/P\lambda^2 = 0.36328\ \text{EIc}/P\lambda^2$$

$$\text{Critical Load Pcr} = 0.36328 \times 36\ \text{EIc}/L^2c = 13.078\ \text{EIc}/L^2c$$

$$12.39\ \text{EIc}/L^2c < \text{Pcr} < 14.52\ \text{EIc}/L^2c$$

Fig. 4 - Calculation of critical load P_{cr} for symmetrical two hinged frame (first cycle)

Cycle Number (2)

y_a	0		-520		-890		-1000		-828		-445		0	
α	0		520		890		1000		828		445		0	P/EIc
$\bar{\alpha}$			508		868		977		810		440		77	Pλ/EIc
Ass. φav		(-1200)		-692		176		1153		1963		2403		Pλ/EIc
Trial y_p	0		-1200		-1892		-1716		-563		1400		3803	Pλ²/EIc
y_c	0		-634		-1268		-1902		-2535		-3169		-3803	Pλ²/EIc
y_p	0		-1834		-3160		-3618		-3098		-1769		0	Pλ²/EIc
$\varphi_p$av												1769		Pλ/EIc
φ_p													1846	Pλ/EIc

From Equation (3-1), X1 = (1846/5)P = 369.2 P

y	0	-1475	-2503	-2787	-2278	-1205	0	Pλ²/EIc
y_a/y		0.35254	0.35557	0.35881	0.36348	0.36929		EIc/Pλ²

Better Ratio = (3683/10248) EIc/Pλ² = 0.35939 EIc/Pλ²

Critical Load Pcr = (0.35939 × 36) EIc/L²c = 12.94 EIc/L²c

12.69 EIc/L²c < Pcr < 13.29 EIc/L²c

Assumed Buckling Mode of Next Cycle:

y_a	0	-530	-898	-1000	-817	-432	0

Fig. 5 - Calculation of critical load P_{cr} for symmetrical two hinged frame (second cycle)

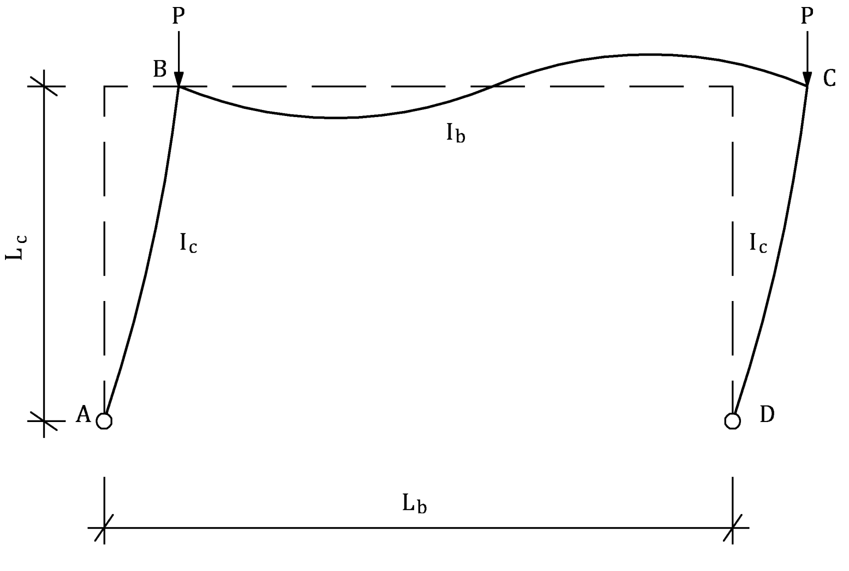


Fig. 6 - Antisymmetric mode of buckling of a two hinged frame

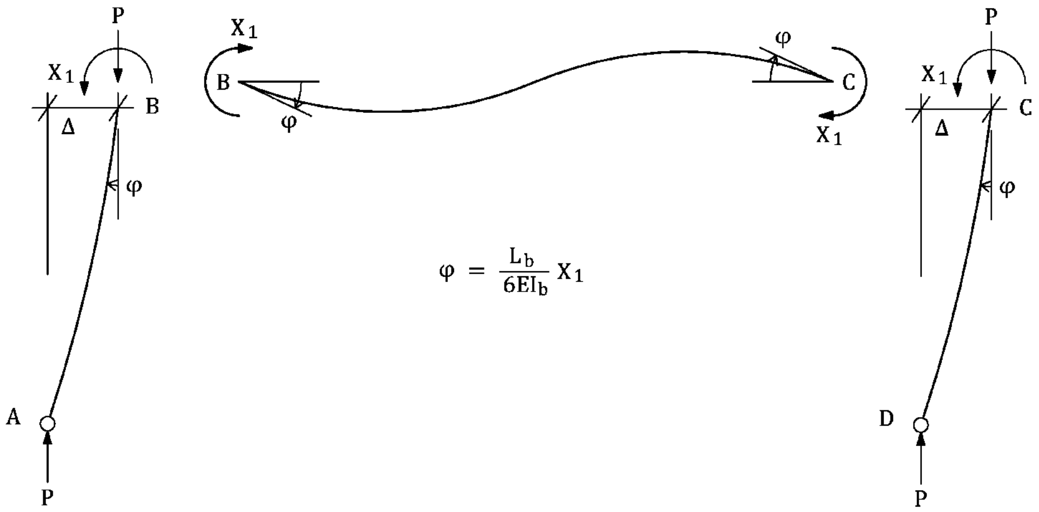


Fig. 7 - Antisymmetric two hinged frame: end forces and rotations

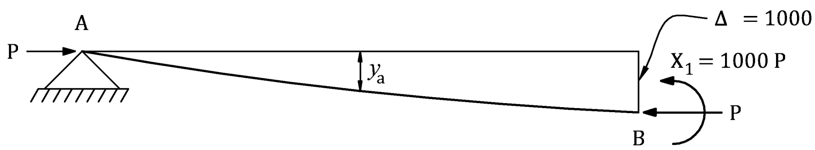
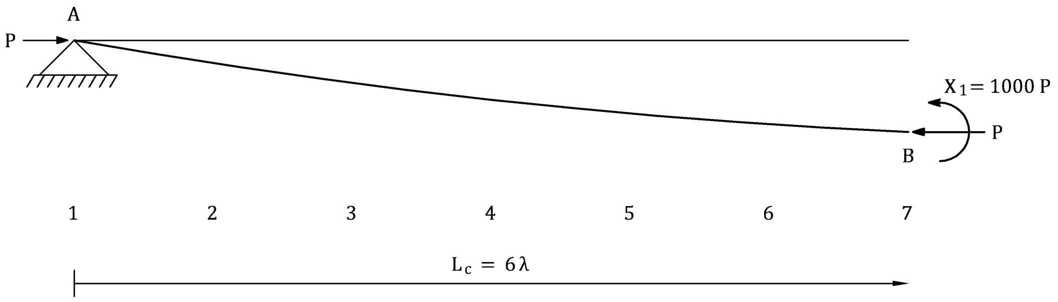


Fig. 8 - Antisymmetric two hinged frame: column AB with an arbitrary sidesway Δ of 1000 units



Cycle Number (1)

1- y_a	0		260		500		707		866		966		1000	
2- $\bar{\alpha}$	0		-258		-497		-703		-861		-960		-497 (1000)	Pλ/EIc
3- φav		4776		4518		4021		3318		2457		1497		Pλ/EIc
4- y	0		4776		9294		13315		16633		19090		20587	$P\lambda^2/EIc$
5- y_a/y			0.05440		0.05380		0.05380		0.05207		0.05060		0.04857	$EIc/P\lambda^2$

Better Ratio = (4299/83695) $EIc/P\lambda^2 = 0.051365\ EIc/P\lambda^2$
Critical Load Pcr = $0.051365 \times 36\ EIc/L^2c = 1.849\ EIc/L^2c$

Cycle Number (2)

y_a	0		232		451		647		808		927		1000	
$\bar{\alpha}$			-231		-449		-644		-805		-923		-490	Pλ/EIc
φav		4542		4311		3862		3218		2413		1490		Pλ/EIc
y	0		4542		8853		12715		15933		18346		19836	$P\lambda^2/EIc$
y_a/y			0.05108		0.05094		0.05088		0.05071		0.05050		0.05041	$EIc/P\lambda^2$

Better Ratio = (4065/80225) $EIc/P\lambda^2 = 0.05067\ EIc/P\lambda^2$
Critical Load Pcr = $0.05067 \times 36\ EIc/L^2c = 1.824\ EIc/L^2c$

Cycle Number (3)

y_a	0		229		446		640		803		925		1000	
$\bar{\alpha}$			-228		-444		-637		-800		-921		-489	Pλ/EIc
φav		4519		4291		3847		3210		2410		1489		Pλ/EIc
y	0		4519		8810		12657		15867		18277		19766	$P\lambda^2/EIc$
y_a/y			0.05067		0.05062		0.05056		0.05061		0.05061		0.05059	$EIc/P\lambda^2$

Better Ratio = (4043/79896) $EIc/P\lambda^2 = 0.05060\ EIc/P\lambda^2$
Critical Load Pcr = $0.05060 \times 36\ EIc/L^2c = 1.8216\ EIc/L^2c$

Cycle Number (4)

y_a	0	228.6	445.7	640.3	802.7	924.6	1000

Fig 9 - Antisymmetric mode of buckling of two hinged frame: complete solution